Analysis of the Problem on the Flow Round a Ball on the Base of Quasi-Hydrodynamic Equations in Stokes Approximation

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1. Introduction.

For investigation of slow isothermal viscous gas (liquid) flows around rigid bodies widely used the Navier--Stokes system in Stokes approximation. Well-known analytical solution of this system is constructed for the problem on the flow round a ball with no-slip boundary conditions. Using this solution the external force applied to ball was calculated [1].

Deviations from Stokes law were observed for slow air flows near small oil drops with size order of mean free path of the molecules. American physicist R. Millikan experimentally established the effect of diminishing the force of resistance, when Knudsen number Kn tends to unity. He also proposed more exact empirical expression for this force [2]. In Knudsen number range [0.01, 0.1] the experimental data may be adequately described by classical Stokes system, if refuse from no-slip conditions on the drop surface and use Maxwell slip conditions. If $Kn \in (0.1,\, 0.5]$, then necessity in construction of alternative mathematical models is arising.

The hierarchy of new mathematical models, based on the so called system of Quasi-Hydrodynamic (QHD) equations, was considered in the monographs [3], [4]. The essential difference from Navier-Stokes theory consisted in using the procedure of spatial-temporal averaging for determination of main macroscopic values -- density, velocity and temperature.

In the report new analytical solution of QHD system in Stokes approximation for the problem on the external flow round a ball with slip boundary conditions will be constructed. More exact expression for force of resistance and some results of numerical calculations will be presented.

2. Statement of the problem.

Stokes approximation of stationary QHD system in spherical coordinates $(r,\ \varphi,\ \theta)$ without influence of external forces has a form

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, u_\theta\right)=\tau\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial p}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial p}{\partial\theta}\right)\right], \tag{1}$$

$$\frac{\partial p}{\partial r}=2\nu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\sigma_{rr}\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\sigma_{\theta r}\right)-\frac{\sigma_{\theta\theta}+\sigma_{\varphi\varphi}}{r}\right], \tag{2}$$

$$\frac{1}{r}\frac{\partial p}{\partial\theta}=2\nu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\sigma_{r\theta}\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\sigma_{\theta\theta}\right)+\frac{\sigma_{\theta r}-\sigma_{\varphi\varphi}\cot\theta}{r}\right], \tag{3}$$

In (1)-(3) the component of velocity u_φ is equal to zero. Other macroscopic parameters are not depending on φ. Connection of spherical coordinates $(r,\ \varphi,\ \theta)$ with Cartesian ones (x,y,z) is given by relations $x=r\cos\varphi\sin\theta$, $y=r\sin\varphi\sin\theta$, $z=r\cos\theta$. Components of velocity strain tensor $\hat{\sigma}$ are calculated by the formulas

$$\sigma_{rr}=\frac{\partial u_r}{\partial r},\quad \sigma_{r\theta}=\sigma_{\theta r}=\frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right),$$

$$\sigma_{\theta\theta}=\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r},\quad \sigma_{\varphi\varphi}=\frac{u_r}{r}+\frac{u_\theta\cot\theta}{r}.$$

Small relaxation parameter τ is determined with the help of expression

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$$\tau = \frac{\gamma}{Sc}\frac{\nu}{c_s^2},$$

in which $\nu = \eta / \rho$ is the coefficient of kinematical viscosity, η is the coefficient of dynamical viscosity, ρ is the density, $c_s = \sqrt{\gamma R_* T}$ is the sonic velocity, T is the temperature, R_* is the gas constant, γ is the specific heat ratio, Sc is the Schmidt number. In particular, for air we have γ=1.4, Sc=0.74.

System (1)-(3) is closed under unknown functions - the components of velocity vector $u_r=u_r(r,\theta)$, $u_\theta=u_\theta(r,\theta)$ and the pressure $p=p(r,\theta)$, divided to constant density ρ. If $\tau \to 0$, then it is transform to classical Stokes system.

The problem of the external uniform flow round a ball of radius R with the center in origin of coordinates, directing along oz axis and having under $r \to +\infty$ velocity U_∞ and pressure p_∞, consist in finding the functions u_r, u_θ and p, satisfying in the domain $G = \{(r,\theta): R < r < +\infty,\ 0 < \theta < \pi\}$ the equations (1)-(3), and the conditions

$$u_r(R,\theta) = 0,\ \ u_r(+\infty,\theta) = U_\infty \cos\theta; \tag{4}$$

$$u_\theta(R,\theta) = \frac{2-\xi}{\xi}\lambda \frac{\partial u_\theta}{\partial r}(R,\theta),\ \ u_\theta(+\infty,\theta) = -U_\infty \sin\theta; \tag{5}$$

$$\frac{\partial p}{\partial r}(R,\theta) = 0,\ \ p(+\infty,\theta) = p_\infty;\ 0 < \theta < \pi. \tag{6}$$

Here ξ is the part of the molecules, reflected by diffusive way, which is approximately equal to unity. For placed in the air oil drop we may put ξ =0.9. Mean free path of the molecules λ is calculated according to well-known [2] D.Chapman formula

$$\lambda = \nu\sqrt{\frac{\pi}{2R_* T}}$$

First equality (5) is the Maxwell slip condition for tangential component of the velocity vector near drop surface.

3. Self--similar solution.

The solution of posed problem will be finding in the form

$$\begin{aligned} u_r &= A\left(\frac{r}{R}\right)U_\infty \cos\theta,\quad u_\theta = -B\left(\frac{r}{R}\right)u_\infty \sin\theta, \\ p &= p_\infty + C\left(\frac{r}{R}\right)U_\infty^2 \cos\theta. \end{aligned} \tag{7}$$

Substitution (7) in (1)-(3) gives

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2 A\right) - \frac{2B}{x} = \frac{\delta^2 \operatorname{Re}}{2}\left[\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{dC}{dx}\right) - \frac{2C}{x^2}\right], \tag{8}$$

$$\frac{dC}{dx} = \frac{1}{\operatorname{Re}}\left[\frac{2}{x^2}\frac{d}{dx}\left(x^2\frac{dA}{dx}\right) - \frac{2}{x}\frac{dB}{dx} - 6\frac{A-B}{x^2}\right], \tag{9}$$

$$C = \frac{1}{\operatorname{Re}}\left[\frac{1}{x}\frac{d}{dx}\left(x^2\frac{dB}{dx}\right) + \frac{dA}{dx} + 4\frac{A-B}{x}\right]. \tag{10}$$

It is necessary to supplement the system with the conditions

$$A(1) = 0,\quad A(+\infty) = 1; \tag{11}$$

$$B(1) = \varsigma\frac{dB}{dx}(1),\quad B(+\infty) = 1; \tag{12}$$

$$\frac{dC}{dx}(1)=0,\quad C(+\infty)=0. \tag{13}$$

Here the Reynolds number is determined with the help of expression $\mathrm{Re}=(U_\infty R)/\nu$. By symbol x the ratio r/R is designated. Connection of positive constants δ and ς with Knudsen number $Kn=\lambda/R$ is given by relations $\delta=\sqrt{2\tau\nu}/R=2Kn/\sqrt{\pi Sc}$, $\varsigma=\alpha\, Kn$, where $\alpha=(2-\xi)/\xi$.

The problem of finding the solution of the system (8)-(10) on the interval $(1,+\infty)$, satisfying the conditions (11)-(13), may be reduced to the problem of integrating the modified Bessel equation. The final result is representing as

$$A(x)=1-\frac{1}{x^3}-c_1^*\left(\frac{1}{x}-\frac{1+\delta^2}{x^3}\right)-\frac{c_2^*}{x^3}\left[2\left(1+\frac{x}{\delta}\right)+\frac{x^2}{\delta^2}\right]\exp\left(-\frac{x}{\delta}\right), \tag{14}$$

$$B(x)=1+\frac{1}{2x^3}-\frac{c_1^*}{2}\left(\frac{1}{x}+\frac{1+\delta^2}{x^3}\right)+\frac{c_2^*}{x^3}\left(1+\frac{x}{\delta}\right)\exp\left(-\frac{x}{\delta}\right), \tag{15}$$

$$C(x)=-\frac{1}{\mathrm{Re}}\left[\frac{c_1^*}{x^2}-\frac{2c_2^*}{x^2\delta^2}\left(1+\frac{x}{\delta}\right)\exp\left(-\frac{x}{\delta}\right)\right], \tag{16}$$

where

$$c_1^*=\frac{3+6\delta+6\delta^2}{2\dfrac{1+2\zeta}{1+\zeta}+4\dfrac{1+2\zeta}{1+\zeta}\left(\delta+\delta^2\right)+\delta^2}, \tag{17}$$

$$c_2^*=\frac{3\delta^4}{2\dfrac{1+2\zeta}{1+\zeta}+4\dfrac{1+2\zeta}{1+\zeta}\left(\delta+\delta^2\right)+\delta^2}\exp\left(\frac{1}{\delta}\right). \tag{18}$$

Turning in (14)-(18) to limit under $\delta\to 0$, $\zeta\to 0$ and substituting the functions $A(x)$, $B(x)$, $C(x)$ to (7), we obtain the classical Stokes solution [1].

4. Force of resistance.

Using constructed solution, for $Kn \le 0.5$ and small Re numbers the external force applied to spherical oil drop and directed along oz axis may be calculated:

$$F_{QHD}^{slip}=2\pi R^2\int_0^\pi\left(-pp\cos\theta+2\eta\sigma_{\tau\tau}\cos\theta-2\eta\sigma_{\theta\tau}\sin\theta\right)\Big|_{r=R}\sin\theta d\theta=$$

$$=\frac{2+4\delta+4\delta^2}{2\dfrac{1+2\zeta}{1+\zeta}+4\dfrac{1+2\zeta}{1+\zeta}\left(\delta+\delta^2\right)+\delta^2}6\pi\eta RU_\infty=$$

$$=\left[1-\alpha Kn+2\left(\alpha^2-\frac{1}{\pi Sc}\right)Kn^2+\ldots\right]6\pi\eta RU_\infty. \tag{19}$$

Formally turning in (19) parameter δ to zero, we obtain corresponding expression for force of resistance, which follows from Navier-Stokes theory when slip Maxwell effects on the drop surface are taking into account:

$$F_{NS}^{slip}=\frac{1+\zeta}{1+2\zeta}6\pi\eta RU_\infty=\left[1-\alpha Kn+2\alpha^2Kn^2+\ldots\right]6\pi\eta RU_\infty. \tag{20}$$

Passage to the limit in (20) under $\zeta\to 0$ leads to classical Stokes law.

Experimental values of the force of resistance may be calculated [2] with the help of Millikan formula

$$F_M=\frac{6\pi\eta RU_\infty}{1+\alpha Kn}=\left[1-\alpha Kn+\alpha^2Kn^2+\ldots\right]6\pi\eta RU_\infty.$$

It is easy to verify, that for ξ=0.9, Sc=0.74 and ≤ 0.5 take place the inequalities

$$F_M(Kn) < F_{QHD}^{slip}(Kn) < F_{NS}^{slip}(Kn).$$

Thus, function $F_{QHD}^{slip}(Kn)$ better than $F_{NS}^{slip}(Kn)$ describes the experimental data in indicated Knudsen number range.

This work is supported by RFBR grant N 01--01--00061.

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