Parallel solution of Stefan problem with prescribed convection

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We consider the Stefan problem with prescribed convection, which the most important application is the continuous casting problem.

The corresponding initial boundary-value problem can be written as

$$\begin{cases} \dfrac{\partial \chi}{\partial t} + \upsilon \dfrac{\partial \chi}{\partial x_2} - \Delta u = 0, \quad \chi(x,t) \in H(u(x,t)) \text{ for } x \in \Omega, t > 0, \\ u = z(x,t) \;\; x \in \Gamma_D,\, t > 0, \quad \dfrac{\partial u}{\partial n} + g(u) = 0 \text{ for } x \in \Gamma_N, t > 0, \\ \chi = H_0(x) \;\text{ for } x \in \bar{\Omega},\, t = 0 \end{cases} \tag{1}$$

with monotone and continuous function $g(.)$, and maximal monotone multi-valued function $H(.)$. Here Γ_D and Γ_N are the parts of the boundary for the domain $\Omega \subset \mathrm{R}^3$, u stands for the temperature of a substance, χ for the enthalpy, while υ is the speed of casting. The existence of a unique solution for problem (1) is proved in [13].

Problem (1) is approximated by the implicit mesh scheme

$$\partial_{\bar{t}}\chi + Au + B\chi + Cu = F, \chi \in H(u), \tag{2}$$

where $\partial_{\bar{t}}\chi$ is the backward approximation of the time derivative $\dfrac{\partial \chi}{\partial t}$, and by so-called characteristic scheme

$$d_{\bar{t}}\chi + Au + Cu = F, \chi \in H(u), \tag{3}$$

where $d_{\bar{t}}\chi$ approximates the first order operator $\dfrac{\partial \chi}{\partial t} + \upsilon \dfrac{\partial \chi}{\partial x_2}$. Above A is a mesh approximation of Laplace operator, $B\chi$ is an up-wind approximation for nonlinear convective term, C is a monotone and continuous operator, and H is a nonlinear multivalued operator. Both mesh schemes (2) and (3) belong to a class of finite-dimensional problems with several multivalued operators

$$Au + \sum_{k=1}^{s} B_k \gamma_k = f; \;\; \gamma_k \in C_k u. \quad \forall k = 1,\ldots,s\,. \tag{4}$$

Problem (4) includes as a partial case the "traditional" finite-dimensional variational inequalities, such as obstacle problem, two-sided obstacle problem etc. (they correspond to s=1 and unit matrix B_1).

We prove the unique solvability of (4) in the case of M-matrices A, B_k and diagonal maximal monotone operators C_k, and the convergence and geometric rate of convergence for several classes of iterative methods. They are, in particularly, multisplitting method (constructed in [10] for linear equations), Schwarz-type methods, based on the domain decomposition with overlapping subdomains (studied in [4], [3], [1], [11], [16], [17] for obstacle-type problems).

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For problem (3) with positive definite matrix A the splitting iterative method in combination with non-overlapping domain decomposition is also investigated theoretically and numerically.

The most attention is paid to parallel implementation of the iterative algorithms and studying their rate of convergence and scalability; numerical comparison of all the algorithms is executed.

Also, several new predictor-corrector mesh schemes for time-dependent problem (1) are constructed and numerically investigated (cf. [14], [15] for theory of so-called regional-additive schemes for linear time-dependent problems, and [12] for predictor-corrector schemes applied to nonlinear equations). These schemes are treated as the splitting ones and the splitting of the operators are made on the basis of the domain decomposition with non-overlapping subdomains. Namely, a variant of this scheme, corresponding to implicit approximation (2) is

$$\begin{cases} \frac{1}{\tau}\left(\chi^{n+1/2}-\chi^{n}\right)+A_1u^{n+1/2}+A_2u^{n}+B\chi^{n+1/2}+Cu^{n+1/2}=F, \\ \chi^{n+1/2}\in H\left(u^{n+1/2}\right), \\ \frac{1}{\tau}\left(\chi^{n+1}-\chi^{n}\right)+A_1u^{n+1/2}+A_2u^{n+1}+B\chi^{n+1}+Cu^{n+1}=F, \\ \chi^{n+1}\in H\left(u^{n+1}\right) \end{cases} \quad (5)$$

with $A_2=\chi A$, $A_1=(1-\chi)$A and χ being the characteristic function for the set of all subdomains boundaries. The implementation of (5) for fixed time level $n+1$ consists of three steps: finding the solutions on the subdomains boundaries (predictor step), concurrent solution of the "subproblems" in the subdomains (main step), and improvement of the solutions on the subdomains boundaries (corrector step). The predictor-corrector schemes were founded to be unconditionally stable and highly scalable.

In the report, we cite the main theoretical results (basically published in [5]-[9]) as well as a number of numerical results for model and real life continuous casting problems. The calculations were executed by using Cedar computer (Espoo, Finland).

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