Numerical Methods and Parallel Algorithms for Study of Burgers' Turbulence

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The Burgers' turbulence (BT) is known as the simplest model of real turbulence [1] and includes its main mechanisms - nonlinear interaction of disturbances, their spectral transfer and viscous dissipation on the small scales. BT is described by Burgers' equation (BE) for the velocity field with appropriate choice of initial conditions and/or external driving forcing:

$$\vec{\mathbf{v}}_{\mathbf{t}} + (\vec{\mathbf{v}}\nabla)\vec{\mathbf{v}} = \nu\Delta\vec{\mathbf{v}} + \vec{\mathbf{f}}(\vec{\mathbf{x}},\mathbf{t}); \quad \vec{\mathbf{v}}(\vec{\mathbf{x}},\mathbf{t}=0) = \vec{\mathbf{v}}_0(\vec{\mathbf{x}}). \tag{1}$$

Here ν is the kinematic viscosity coefficient.

The properties of BT are of great interest in many aspects: for the general theory of nonlinear partial differential equations, for development of methods of their effective numerical solution, for the statistical theory of turbulence, as well as for numerous applications in various scientific fields, including such problems as the formation of the large-scale structure in the Universe, nonlinear acoustics and optics, formation of structures on the surface in inhomogeneously heated fluid, interface growth, traffic current etc. [1, 2]. In spite of the fact that Hopf-Cole transform gives the solution of BE directly for the current time, its solution using time forwarding is also of great interest, as a good test for developing effective numerical methods for wider class of nonlinear equations with sharp gradients.

In the present report few numerical methods are described, which have been implemented for solution of BT problems with random self-similar initial conditions. The realization of parallel versions of these methods on multiprocessor systems of SIMD and MIMD classes is discussed. It is assumed that external force is absent, $\vec{\mathbf{f}} = 0$, and the initial velocity is determined by scaling exponent *h* [3, 4]:

$$v_0(x'+\lambda x) - v_0(x') = \lambda^h (v_0(x'+x) - v_0(x')) \quad \forall\lambda > 0, \forall x, \forall x',$$

where the equality is valid in statistical sense, i.e. the probability distribution of both sides of equality is the same. Values $-1 \le h \le 1$ correspond to random functions of different types, particularly self-similar 1D signal is Brownian motion for $h = 1/2$, and white noise for $h = -1/2$. The solution of (1) without external forcing ($\vec{\mathbf{f}} = 0$) at vanishing viscosity reads:

$$\vec{\mathbf{v}}(\vec{\mathbf{x}},\mathbf{t}) = \frac{\vec{\mathbf{x}} - \vec{\mathbf{a}}(\vec{\mathbf{x}},\mathbf{t})}{\mathbf{t}},$$

where $\vec{\mathbf{a}}(\vec{\mathbf{x}},\mathbf{t})$ is the inverse Lagrangian function, defined as the coordinate where the maximum is achieved for the function

$$\mathbf{G}(\vec{\mathbf{x}},\vec{\mathbf{a}},\mathbf{t}) = \boldsymbol{\Psi}_0(\vec{\mathbf{a}}) - \frac{(\vec{\mathbf{x}} - \vec{\mathbf{a}})^2}{2\mathbf{t}}. \tag{2}$$

The numerical determination of function $\vec{\mathbf{a}}(\vec{\mathbf{x}},\mathbf{t})$ in the inviscid limit is based on the possibility to formulate the solution of (1) at $\nu \to 0$ for the velocity potential $\boldsymbol{\Psi}(\vec{\mathbf{x}},\mathbf{t})$ in terms of Legendre Transform:

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$$\mathbf{\Psi}(\vec{\mathbf{x}},\mathbf{t}) = \max_{\vec{\mathbf{a}}}\left[\mathbf{\Psi}_0(\vec{\mathbf{a}},\mathbf{t}) - \frac{(\vec{\mathbf{x}}-\vec{\mathbf{a}})^2}{2\mathbf{t}}\right], \tag{4}$$

$$\mathbf{H}(\vec{\mathbf{x}},\mathbf{t}) = \max_{\vec{\mathbf{a}}}\left[\mathbf{\Phi}_0(\vec{\mathbf{a}},\mathbf{t}) + \frac{(\vec{\mathbf{x}}\vec{\mathbf{a}})}{2\mathbf{t}}\right]. \tag{5}$$

Here $\mathbf{H}(\vec{\mathbf{x}},\mathbf{t}) = \mathbf{\Psi}(\vec{\mathbf{x}},\mathbf{t}) + \frac{(\vec{\mathbf{x}}^2)}{2\mathbf{t}}$; $\mathbf{\Phi}_0(\vec{\mathbf{a}},\mathbf{t}) = \mathbf{\Psi}_0(\vec{\mathbf{a}},\mathbf{t}) - \frac{(\vec{\mathbf{a}}^2)}{2\mathbf{t}}$ and the right hand side of (5) is the Legendre transform of $\mathbf{\Phi}_0(\vec{\mathbf{a}},\mathbf{t})$.

For the numerical solution of Burgers' equation in the inviscid limit recently developed algorithm of Fast Legendre Transform (FLT) is used, realised in both variants for sequential one-processor computers [4] and for parallel multiprocessor SIMD supercomputers [5]. The FLT algorithm is based on the important property of the inverse Lagrangian function, which determines the solution of Burgers' equation in the inviscid limit [4]:

$$((\vec{\mathbf{a}}(\vec{\mathbf{x}},\mathbf{t}) - \vec{\mathbf{a}}(\vec{\mathbf{x}}',\mathbf{t}))(\vec{\mathbf{x}} - \vec{\mathbf{x}}')) \geq 0. \tag{6}$$

In the 1-D case (6) is equivalent to the monotonic (non-decreasing) dependence of the inverse Lagrangian function from the space coordinate, and in the multidimensional case it means, in particular, that each component of this function is a non-decreasing function of the corresponding coordinate.

When realised on the Connection Machine CM200 parallel SIMD computer, the parallel FLT algorithm was based on binary tree parallel search of maximum in (4), using "send" and "segmented scan" operations. The estimate of computation speedup in an ideal case of hypothetical parallel SIMD computer with number of processors equal to number of mesh points N as $S_{SIMD} \sim \frac{N}{\log_2 N}$, and in a real case of computer with N_{phys}^{SIMD} processors we get $S_{SIMD}^{phys} \sim \frac{N_{phys}^{SIMD}}{\log_2 N}$. In the present study another form of this parallel algorithm is developed, suitable for the MIMD type parallel computers and realized on 16-processor parallel cluster MVS-1000. For the common case of MIMD computer with N_{phys}^{MIMD} processors we yield the estimate of the speedup as

$$S_{MIMD}^{phys} \sim \frac{N_{phys}^{MIMD}}{\log_2(N/N_{phys}^{MIMD})} \frac{\log_2 N}{\log_2 (N_{phys}^{MIMD})^2}.$$

The ratio of speedups on SIMD and MIMD computers is therefore

$$r \sim \frac{N_{phys}^{SIMD}}{N_{phys}^{MIMD}} \frac{(\log_2 N_{phys}^{MIMD})^2}{\log_2 N},$$

and we have $r > 1$ for reasonable values of parameters. For instance, for the cases considered $N = 2^{20}$; $N_{phys}^{SIMD} = 8192$ for CM 200, and $N_{phys}^{MIMD} = 16$ for MVS-1000 we could obtain $S_{SIMD}^{phys} \sim 410$ and $S_{MIMD}^{phys} \sim 1.25$, so $r \sim 300$, and it is required about 300 times growth of CPU performance to reach on the MVS-1000 system the computational time of FLT algorithm comparable with that reached at CM200.

In the case of nonzero viscosity one of the most interesting and promising modern methods of numerical solution of nonlinear partial differential equations is using of wavelet bases.

The decay of the wavelet coefficients depends upon the local smoothness of the function. Therefore if function is smooth everywhere except several regions of sharp transition (``singularities"), then the wavelet coefficients are sufficiently large only in the vicinity of the

singularities of the function. It allows decreasing number of the coefficients to represent solutions with sharp transitions.

For the broad class of differential and pseudo-differential operators, in particular for Laplace operator, contained in BE, wavelet bases yield the sparse block-band structure of their representation.

Another advantage of the wavelet expansion is enough simple implementation of the dynamical adaptation of the space resolution, i.e. automatic following singular structures of the solution and increasing resolution in their vicinities [9].

Moreover wavelets allow to implement efficient parallel algorithms of the numerical integration of evolution equations, in particularly to compute the parallel multiplication of a matrix by a vector [8]. To do the parallel calculations we separate each scale (represented by a $N = 2^n$ points) in 2^{j_p} parts of equal size. If we have N points and $N_p = 2^{j_p}$ processes, the computation complexity is $O(\log_2(N) \cdot N / N_p)$. Accounting that complexity of this algorithm on one-processor system is $O(N^2)$ the estimate of speedup is $O(N_p N / \log_2 N)$.

For the numerical integration of BE we used compactly supported Daubechies wavelets [7]. Moreover we develop biorthogonal compactly supported wavelet 1D and 2D bases adapted to linear differential operators (in particular, for Laplace operator), yielding their diagonal representation, as for the Fourier basis.

Numerical solution of BE is fulfilled for initial conditions with $(-0.75 \le h \le 0.75)$ up to the time characterized by formation and interaction between multiple discontinuities. Realizations with lengths up to $2^{17} \div 2^{20}$ points have been used and the structure functions of velocity, averaged over several decades and few hundreds of realizations are calculated. Structure function of the velocity field $u(x)$ of order q and its scale invariance in the inertial range, expressed by power law, can be written as follows

$$S_q(l) = <\delta u_l^q> = \frac{1}{L}\int_0^L |u(x+l) - u(x)|^q \, dx = C_q l^{\zeta_q} \quad \text{for all } q, \tag{7}$$

where C_q are some numerical constants, ζ_q are scaling exponents.

The use of parallel forms of Fast Legendre Transform algorithm in case of vanishing viscosity and wavelet bases for non-zero viscosity allows to decrease sufficiently computational time for gathering required statistics, and to made it possible to investigate self-similar properties of $S_q(l)$ in BT. Particularly, we have checked the existence in BT the important property of extended self-similarity (usually referring as ESS) which is found recently and investigated theoretically and experimentally for rather wide range of real 3D turbulent flows, including isotropic 3D turbulence, thermal convection and MHD-turbulence [3]. ESS allows to determine with significantly improved accuracy the scaling exponents ζ_q due to extension of the interval in which power laws obey. The existence of some kind of extended interval of scaling behaviour for velocity structure function in BT was confirmed numerically for both viscous and inviscid cases for the medium and large scales. More detailed study indicates however that in the viscous subrange the ESS property possibly not hold. This violation of ESS become more significant, and is more expressed for random initial conditions of fractal Brownian type ($h \ge 0$), and relatively weak for fractal white noise initial conditions ($h \le 0$). These numerical results could mean that existence of ESS is not universal for any kind of random process, but is somehow connected with the mechanism of the energy dissipation, differing sufficiently in Navier-Stokes and Burgers' turbulence. The results indicate that for the case of BT the analysis of ESS for 3D Navier Stokes turbulence [10] seems to be also valid, giving more expressed deviation due to more strong intermittency of BT as compared with isotropic Navier-Stokes turbulence. It could be measured by the deviation from Kolmogorov law of scaling exponent for second structure

function of velocity, which is relatively small in the last case, $\delta\zeta \approx 0.03$, and for BT is much greater and depends sufficiently from h [3, 4].

Thus, the presented methods, implemented for investigation of statistical and scaling properties of Burgers' turbulence, are useful tools, promising for generalizations to other nonlinear amplitude equations, which haven't generally simplified alternative methods of solution, like Hopf-Cole transform. When used for the solution of the BT problem, these methods allow to reach an effective resolution of the regions of sharp transitions. In the inviscid limit using FLT algorithm, in parallel variants in the extreme, allows to reduce sufficiently the computation time and gives the possibility to gather enough large statistics for Burgers' turbulence. We could see also, that for parallel form of the FLT algorithm using SIMD computers with enough large number of processors is preferable. In the case of finite viscosity, wavelet bases allow to implement the algorithm combined all advantages of the such representation: the nonlinear compression of the solution, sparse representation of the differential operators and dynamical adaptation of the space resolution. In comparison with traditional spectral and finite difference methods, wavelet bases allow to decrease essentially the number of the degrees of freedom that is needed for obtaining the solution with a given accuracy.

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References

1. Gurbatov S.N., Malakhov A.N., Saichev A.I. Nonlinear random waves and turbulence in nondispersive media: waves, rays, particles. Manchester University Press, 1991.
2. Whitham G.B. Linear and nonlinear waves. Wiley, 1974.
3. Frisch U. Turbulence: the legacy of A.N. Kolmogorov.- Cambridge University Press, 1995
4. She Z.S., Aurell E., Frisch U. The inviscid Burgers' equation with initial data of Brownian type. Comm. Math. Phys., V. 148, P. 623-641, 1992.
5. Aurell E., Wertgeim I.I. Fast Parallel Legendre Transforms. ``Science on the Connection Machine System", Cambridge (Massachusetts, USA): Thinking Machines Corporation, P.~121--128, 1995.
6. Bacry E., Mallat S., Papanicolaou G. A wavelet based space-time adaptive numerical method for partial differential equations. M^2AN, V. 26, No. 7, P. 793-834, 1992.
7. Daubechies I. Ten lectures on wavelet. SIAM, Philadelphia, 1992.
8. Goiran E., Wickerhauser M.V. A parallel two dimensional wavelet packet transform and its application to matrix-vector multiplication. Wavelet Applications in Chemical Engineering, Rodolphe L. Motard and Babu Joseph Eds., Kluwer Academic Publishers, Norwell, Massachusetts, P. 275-319, 1994.
9. Liandrat J, Tchamitchian Ph. Resolution of the 1D Burgers equation using a spatial wavelet approximation algorithm and numerical results. Technical report, ICASE, 1990.
10. Fujisaka H., Grossmann S. Scaling hypothesis leading to extended self-similarity in turbulence. *Physical Review E*, 63:026305, 2001.