Parallel Version of the Iterative Space-Marching Method for Navier-Stokes equations

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The iterative space-marching method (IMM) for incompressible and compressible Navier-Stokes equations is developed in works [1-4] and other. This method is applied to solve steady and unsteady, internal and external 1D, 2D, 3D problems. The main characteristic feature of the method is the use a common procedure for solving various types of problems, namely: marching on one coordinate for 1D, 2D problems and on two coordinates for 3D problems. Thus a single algebraic procedure is used (repeatedly) for all problems. This is the procedure of finding the vector of unknown grid functions along transverse line. It is analytically proved that stability and convergence of the method are unconditional for hydrodynamics problems [1, 2]. Good stability and convergence of the method for solving compressible flows are demonstrated on numerical examples [2, 3].

In this work we present computational schemes (in IMM framework) that suitable for solving steady and unsteady problems with parallel algorithm. We prove that they have unconditional convergence. Based on these schemes one processor may be used for computations of the mentioned above algebraic procedure in each marching station on any time layer or global iteration.

First consider the scheme for steady problems. The modified scheme of IMM is efficient for these problems. This scheme is based on principle of convergence in time to a steady state. According to this scheme each time-step is realized with one space marching sweep. In order to formulate a scheme that is realized with a parallel algorithm we change finite-difference approximations of derivatives from velocity components with respect to marching coordinates.

Consider unsteady form of incompressible Navier-Stokes equations system in cartesian coordinates. For simplicity consider 2D equations system. We write her finite-difference analog with "frozen" coefficients in the form

$$\frac{u_{jm}^{\sigma}-u_{jm-1}^{\sigma-1}}{\Delta x}+\frac{v_{j+1m}^{\sigma}-v_{j-1m}^{\sigma}}{2\Delta y}=0,$$

$$\frac{u_{jm}^{\sigma}-u_{jm-1}^{\sigma}}{\Delta t}+|u_0|\frac{u_{jm}^{\sigma}-S_1u_{jm-1}^{\sigma-1}-S_2u_{jm-1}^{\sigma+1}}{\Delta x}+v_0\frac{u_{j+1m}^{\sigma}-u_{j-1m}^{\sigma}}{2\Delta y}+$$

$$+\frac{p_{jm-1}^{\sigma+1}-p_{jm}^{\sigma}}{\Delta x}=\frac{u_{j+1m}^{\sigma}-2u_{jm}^{\sigma}+u_{j-1m}^{\sigma}}{Re\Delta y^2}+\frac{u_{jm-1}^{\sigma+1}-2u_{jm}^{\sigma}+u_{jm-1}^{\sigma-1}}{Re\Delta x^2},$$

$$\frac{v_{jm}^{\sigma}-v_{jm-1}^{\sigma}}{\Delta t}+|u_0|\frac{v_{jm}^{\sigma}-S_1v_{jm-1}^{\sigma-1}-S_2v_{jm-1}^{\sigma+1}}{\Delta x}+v_0\frac{v_{j+1m}^{\sigma}-v_{j-1m}^{\sigma}}{2\Delta y}+$$

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$$+\frac{p^{\sigma}_{j+1m}-p^{\sigma}_{j-1m}}{2\Delta y}=\frac{v^{\sigma}_{j+1m}-2v^{\sigma}_{jm}+v^{\sigma}_{j-1m}}{Re\Delta y^2}+\frac{v^{\sigma+1}_{jm-1}-2v^{\sigma}_{jm}+v^{\sigma-1}_{jm-1}}{Re\Delta x^2}, \quad (1)$$

$$S_1=(1+u_0/|u_0|)/2, \quad S_2=(1-u_0/|u_0|)/2,$$

where indexes σ, j - indexes of the points along marching x direction and transverse y one respectively, m - index of temporary layer, u, v - projections of the velocity vector on axes x, y respectively, p - pressure and Re - Reynolds number.

The scheme (1) differs from the modified scheme considered in [1] in that values of grid functions with index $\sigma-1$ have index $m-1$ (not m as in [1]). Modified scheme is realized with a marching sweep at each temporary layer, i.e. by means of finding solutions successively at lines $\sigma=2,3,...,N-1$. As opposed to that, finding solutions for system (1) at these lines does not require a specific order of steps and therefore may be achieved with a parallel algorithm. However, it is important to investigate whether this system is time convergent. Consider this question.

Let values of grid functions at $m\to\infty$ are $u^{\sigma}_{j\infty}, v^{\sigma}_{j\infty}, p^{\sigma}_{j\infty}$. They have to satisfy the system (1). Then residuals

$$\tilde{u}^{\sigma}_{jm}=u^{\sigma}_{j\infty}-u^{\sigma}_{jm}, \quad \tilde{v}^{\sigma}_{jm}=v^{\sigma}_{j\infty}-v^{\sigma}_{jm}, \quad \tilde{p}^{\sigma}_{jm}=p^{\sigma}_{j\infty}-p^{\sigma}_{jm}$$

also have to satisfy system (1), in which u, v, p are substituted for $\tilde{u},\tilde{v},\tilde{p}$ respectively. Determine the vector of residuals as follows

$$\vartheta^{\sigma}_{jm}=(\tilde{u}_*,\tilde{v}_*,\tilde{p}_*)^T\lambda^m\exp[i(\gamma\sigma+\alpha j)].$$

The matrix of the equations system for $\tilde{u}_*,\tilde{v}_*,\tilde{p}_*$ is

$$\begin{pmatrix} \lambda-e^{-i\gamma} & \lambda i r_y\sin\alpha & 0 \\ \lambda R_1-R_2 & 0 & e^{-i\gamma}-\lambda \\ 0 & \lambda R_1-R_2 & \lambda i r_y\sin\alpha \end{pmatrix}, \quad (2)$$

where

$$R_1=r_t+|u_0|+2/(Re\Delta x)+2r_y(1-\cos\alpha)/(Re\Delta y)+v_0 r_y i\sin\alpha,$$

$$R_2=r_t+|u_0|(S_2e^{i\gamma}+S_1e^{-i\gamma})+(e^{i\gamma}+e^{-i\gamma})/(Re\Delta x), \quad r_t=\Delta x/\Delta t, \quad r_y=\Delta y/\Delta t.$$

The characteristic equation of this matrix is

$$(\lambda-e^{-i\gamma})(\lambda-e^{i\gamma})(\lambda R_1-R_2)+(\lambda R_1-R_2)\lambda^2A^2=0, \quad A^2=r_y^2\sin^2\alpha.$$

It has three roots:

$$\lambda_1=\frac{R_1}{R_2}, \quad \lambda_{2,3}=\frac{\cos\gamma}{1+A^2}\pm\sqrt{\frac{\cos^2\gamma}{(1+A^2)^2}-\frac{1}{1+A^2}}. \quad (3)$$

Consider absolute value of first root. Define: $2r_y(1-\cos\alpha)/(Re\Delta y)=\xi$, then

$$|\lambda_1|^2=\frac{|r_t+|u_0|(S_2+S_1)\cos\gamma+2/(Re\Delta x)\cos\gamma+i|u_0|(S_2-S_1)\sin\gamma|^2}{|r_t+|u_0|+2/(Re\Delta x)+\xi+v_0r_yi\sin\alpha|^2}=$$

$$=\frac{(r_t+2/(Re\Delta x)\cos\gamma+|u_0|\cos\gamma)^2+|u_0|^2\sin^2\gamma}{(r_t+2/(Re\Delta x)+\xi+|u_0|)^2+v_0^2A^2}=$$

$$=\frac{[r_t+2/(Re\Delta x)\cos\gamma]^2+2[r_t+2/(Re\Delta x)\cos\gamma]|u_0|\cos\gamma+|u_0|^2}{[r_t+2/(Re\Delta x)+\xi]^2+2[r_t+2/(Re\Delta x)+\xi]|u_0|+|u_0|^2+v_0^2A^2}.$$

Because $\xi\geq 0, r_t>0$

$$[r_t + 2/(Re\Delta x) + \xi] - [r_t + 2/(Re\Delta x)\cos\gamma] \geq 0$$

and therefore $|\lambda_1| \leq 1$.

Consider the following two roots:

$$|\lambda_{2,3}| = \frac{|\cos\gamma \pm i\sqrt{1 + A^2 - \cos^2\gamma}\,|}{1 + A^2} = \frac{1}{\sqrt{1 + A^2}} \leq 1 .$$

Thus all roots do not exceed 1 therefore scheme (1) converges unconditionally.

The fact of convergence has been confirmed on solving of test problems using conservative form of equations.

Unsteady problems are solved using general scheme of the IMM [1, 2]. In this case the solution at each temporary layer is found using global iterations (GIs). Each GIs is a space-marching sweep. Let us make similar changes in the general scheme to the changes we made above for the modified scheme. Then we would have the following scheme

$$\frac{u^{\sigma}_{jm} - (u^{\sigma-1}_{jm-1})^{s-1}}{\Delta x} + \frac{v^{\sigma}_{j+1m} - v^{\sigma}_{j-1m}}{2\Delta y} = 0 .$$

$$\frac{u^{\sigma}_{jm} - u^{\sigma}_{jm-1}}{\Delta t} + |u_0| \frac{u^{\sigma}_{jm} - S_1(u^{\sigma-1}_{jm})^{s-1} - S_2(u^{\sigma+1}_{jm})^{s-1}}{\Delta x} + v_0 \frac{u^{\sigma}_{j+1m} - u^{\sigma}_{j-1m}}{2\Delta y} +$$

$$+ \frac{(p^{\sigma+1}_{jm})^{s-1} - p^{\sigma}_{jm}}{\Delta x} = \frac{u^{\sigma}_{j+1m} - 2u^{\sigma}_{jm} + u^{\sigma}_{j-1m}}{Re\Delta y^2} + \frac{(u^{\sigma+1}_{jm})^{s-1} - 2u^{\sigma}_{jm} + (u^{\sigma-1}_{jm})^{s-1}}{Re\Delta x^2} ,$$

$$\frac{v^{\sigma}_{jm} - v^{\sigma}_{jm-1}}{\Delta t} + |u_0| \frac{v^{\sigma}_{jm} - S_1(v^{\sigma-1}_{jm})^{s-1} - S_2(v^{\sigma+1}_{jm})^{s-1}}{\Delta x} + v_0 \frac{v^{\sigma}_{j+1m} - v^{\sigma}_{j-1m}}{2\Delta y} +$$

$$+ \frac{p^{\sigma}_{j+1m} - p^{\sigma}_{j-1m}}{2\Delta y} = \frac{v^{\sigma}_{j+1m} - 2v^{\sigma}_{jm} + v^{\sigma}_{j-1m}}{Re\Delta y^2} + \frac{(v^{\sigma+1}_{jm})^{s-1} - 2v^{\sigma}_{jm} + (v^{\sigma-1}_{jm})^{s-1}}{Re\Delta x^2} , \qquad (4)$$

where s - GI number and it is assumed that all unknown functions are related to the current s GI with the exception of those that have index s-1.

The scheme (4) differs from the general scheme considered in [1] in that the values of grid functions with index $\sigma - 1$ have index s-1 (not s as in [1]). Therefore obtaining solutions at transverse lines may be achieved for each GI by means of a parallel algorithm.

Let us investigate convergence of the GIs. Determine the vector of residuals on a current GI

$$(\tilde{u}^{\sigma}_{jm})^s = (u^{\sigma}_{jm})^{\infty} - (u^{\sigma}_{jm})^s, \quad (\tilde{v}^{\sigma}_{jm})^s = (v^{\sigma}_{jm})^{\infty} - (v^{\sigma}_{jm})^s, \quad (\tilde{p}^{\sigma}_{jm})^s = (p^{\sigma}_{jm})^{\infty} - (p^{\sigma}_{jm})^s$$

as follows:

$$(\vartheta^{\sigma}_{jm})^s = (\tilde{u}_*, \tilde{v}_*, \tilde{p}_*)^T \lambda^s \exp[i(\gamma\sigma + \alpha j)] .$$

This vector has to satisfy the system that follows from (4) taking into consideration that values with index m-1 is fixed.

The matrix of the system for $\tilde{u}_*, \tilde{v}_*, \tilde{p}_*$ has the form (2) where expression for R_2 is changed. Namely, the positive term r_t in this expression is absent. Thus we have the same three roots (3) in this case. Each root does not exceed one. Therefore scheme (4) converges unconditionally.

The same results have been obtained for 3D equations system.

Parallel schemes for cases of approximations with second order of marching derivatives [2] and for compressible Navier-Stokes equations [3] may be formulated (in IMM framework) in a similar way.

References

1. Skurin L.I. Iterative Space-Marching Method for Solving Fluid Mechanics Problems. Mathematical Modelling. RAS. 2000. V. 12. N 6. P. 88 - 94.
2. Skurin L.I. Iterative Space-Marching Method for Incompressible and Compressible Full Navier-Stokes Equations. N. Satofuka (Ed.). Proceedings of the First International Conference on Computational Fluid Dynamics, ICCFD, Kyoto, Japan. 10 - 14 July 2000. P. 319 - 324.
3. Polyansky A.P., Skurin L.I. Numerical Modelling of Viscous Sub- and Supersonic Gas Flows with Use of Iterative Space-Marching Method. St. Petersburg Univ. Mech. Bulletin. 1998. N 3. P. 111 - 115.
4. Polyansky A.P., Skurin L.I. Numerical Investigation of 3D Hydrodynamics Problems Using Iterative Space-Marching Method. St. Petersburg Univ. Mech. Bulletin. 1999. N 2. P. 90 - 96.