Chapter 10

Quadratic forms and metrics

Introduction

As the perspicacious reader will have already noticed in the presentation of the main governed mesh generation methods (Chapters 5 to 7) and as will also be seen in the chapters devoted to curve and surface meshing (Chapters 14 and 15), as well as in the sections dealing with h, p and hp-methods (Chapters 21 and 22), lengths, distances and other metric-like relations play an important role and are key features in numerous mesh generation and evaluation algorithms.

From a mathematical point of view, the definition of the length of a given vector (resp. a segment) or, similarly, of the distance between two points is based on an adequate definition of the dot product. Algebraic results indicate that this product is related to a quadratic form (associated with a bilinear form). Depending on the objectives, various definitions of these notions can be exhibited, leading therefore to various definitions of a *metric*.

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This chapter begins with some elementary reviews of quadratic forms (which can be found in classical books), then the notion of length and metric are introduced and explained. The definition of the *unit length* is introduced and discussed in detail as a simple way to measure the length of a given item (segment, vector, etc.) with respect to a given metric. Examples of metrics are given to emphasize the different types of control that can be applied based on the previous notions.

Then, different metric-related operators are suggested. They allow us to apply the various metric manipulations usually involved in a mesh generation or mesh modification context. To this end, we briefly discuss the simultaneous reduction, the interpolation and the intersection of two given quadratic forms. Then, we focus on the metric smoothing problem when these metrics present discontinuities or variations that are too great.

Finally, we briefly discuss a way of constructing metrics suitable for surface meshing and numerical simulations based on finite element methods with a control of the error (of interpolation, for instance).

10.1 Bilinear and quadratic forms

The goal of this first section is to recall some classical definitions and mathematical results of linear algebra, related to linear, bilinear and quadratic forms. These results will notably serve to compute the edge lengths of the given meshes.

10.1.1 Linear and bilinear forms

Let E be a vector space on a field K. We recall that a *basis* of E is a part $a = (a_1, a_2, ..., a_n)$ of E such that each vector $u = (u_1, ..., u_n)$ of E can be written in a unique fashion as :

$$u = \sum_{i=1}^{n} a_i\, u_i\,.$$

All bases of E have the same number of elements, the so-called *dimensions* of E. Now, let consider a vector space E on a commutative field K of characteristics different from 2.

Definition 10.1 *A linear application f, defined on E with value in K, is called a* linear form.

Each linear form has the two following properties :

$$\begin{cases} f(u+v) &= f(u) + f(v) \quad \forall u, v \in E \\ f(\lambda u) &= \lambda f(u) \quad \forall u \in E, \forall \lambda \in K \end{cases}$$

The set $\mathcal{L}(E, K)$ of the all linear applications from E to K is an *additive group* by defining, if f and $g \in \mathcal{L}(E, K)$, the *sum* $f + g$ and the *opposite* $-f$ as :

$$(f+g)(u) = f(u) + g(u) \quad \forall u \in E$$

$$(-f)(u) = -f(u) \quad \forall u \in E\,.$$

Definition 10.2 *We call* bilinear form *on $E \times F$ any bilinear application f from $E \times F$ to K satisfying the two following conditions, $\forall u_j \in E$, $\forall v_j \in F$:*

1. $f(\lambda_1 u_1 + \lambda_2 u_2, v) = \lambda_1 f(u_1, v) + \lambda_2 f(u_2, v), \quad \forall \lambda_j \in K,$
2. $f(u, \mu_1 v_1 + \mu_2 v_2) = \mu_1 f(u, v_1) + \mu_2 f(u, v_2), \quad \forall \mu_j \in K.$

In other words, a bilinear form on $E \times F$ is an application f from $E \times F$ to K which is linear on K in each of its parameters u and v when the other is fixed. The set of bilinear forms on $E \times F$ is a sub-space of K, denoted as $\mathcal{L}_2(E, F)$.

An example of such a bilinear form extensively used in our context is the dot product, defined as the bilinear form f on $(\mathbb{R}^d, \mathbb{R}^d)$ to $\mathbb{R}$:

$$f(u, v) = \langle u, v \rangle = \sum_{k=1}^{d} u_k\, v_k\,,$$

with u_k (resp. v_k) stands for the k^{th} component of vector u (resp. v).

Definitions. Two vectors $u \in E$ and $v \in F$ are said to be *orthogonal* and we write $u \perp v$ if and only if $f(u, v) = 0$. When $E = F$, a bilinear form f is so-called, $\forall (u, v) \in E^2$:

- *symmetric* if and only if : $f(u, v) = f(v, u)$,
- *antisymmetric* if and only if : $f(u, v) = -f(v, u)$,
- *alternate* if and only if : $f(u, u) = 0$,
- *definite symmetric* if and only if : $f(u, u) = 0 \Longleftrightarrow u = 0$.

Let f be a symmetric bilinear form on E. A family $(u_i)_{i \in I}$ (where I is a set of indices) of vectors of E is said to be, $\forall (i, j) \in I^2$:

- *orthogonal*, if and only if :

$$(i \neq j) \Longrightarrow f(u_i, u_j) = 0\,,$$

- *orthonormal*, if and only if :

$$f(u_i, u_j) = \delta_{ij} \quad \text{(Kronecker's symbol, } \delta_{ij} = 0\,, \delta_{ii} = 1\text{)}\,.$$

10.1.2 Matrix form of a bilinear form

Let $a = (a_1, ..., a_n)$ and $b = (b_1, ..., b_p)$ be two bases of E and F, then, we can write, for $i \in [1, n]$, $j \in [1, p]$:

$$f(u, v) = f\Big(\sum_i a_i u_i, \sum_j b_j v_j\Big) = \sum_{i,j} f(a_i, b_j) u_i v_j\,. \tag{10.1}$$

Definition 10.3 *We call (representative) matrix of the bilinear form f of $\mathcal{L}_2(E, F)$ in the basis a and b, the matrix $\mathcal{M} = [m_{ij}]$ determined by :*

$$m_{ij} = f(a_i, b_j)$$

If U and V are two column-matrices of the components of the vectors $u \in E$ and $v \in F$ in the basis a and b, the Relation (10.1) can be expressed as a product of matrices, $\forall (u, v) \in E \times F$:

$$f(u, v) = f(aU, bV) =^t U f(a, b) V = {}^t U\, \mathcal{M}\, V\,. \tag{10.2}$$

$\mathcal{M}$ is *non-degenerate* if and only if $\mathcal{M} = [m_{ij}]$ is *invertible.*

Definition 10.4 *A basis of E is orthogonal (resp.* orthonormal*) if and only if $\mathcal{M}$ is diagonal (resp. equal to the unit matrix of order n).*

10.1.3 Quadratic forms

Definition 10.5 *A quadratic form q on a space vector E is an application from E to its field K such that for each $\alpha \in K$ and $(u,v) \in E^2$:*

1. $q(\lambda u) = \lambda^2 q(u)$,
2. *the application F from $E \times E$ to K, $(u,v) \longrightarrow F(u,v) = q(u+v) - q(u) - q(v)$ is bilinear*[1].

If q_1 and q_2 are two quadratic forms on E and if α_1 and $\alpha_2 \in K$, the application $u \longrightarrow \alpha_1 q_1(u) + \alpha_2 q_2(u)$ is also a quadratic form. Therefore, the set $\mathcal{Q}(E)$ of all quadratic forms on E has a structure of vector space.

Let f be an arbitrary bilinear form on E. The function $q(u) = f(u,u)$ is a quadratic form on E. The sole data of q allows to retrieve the symmetrical part of f :

$$q_f(u+v) - q_f(u) - q_f(v) = f(u,v) + f(v,u)\,.$$

If $K = \mathbb{R}$, the form F can be replaced by the symmetric bilinear form $f = \frac{1}{2}F$. Thus, we have the following relation :

$$f(u,v) = \frac{1}{2}\left(q(u+v) - q(u) - q(v)\right)\,, \tag{10.3}$$

with $f(u,u) = q(u)$. Therefore, the data of a quadratic form q is equivalent to the data of a symmetric bilinear form $f = \frac{1}{2}\left(q(u+v) - q(u) - q(v)\right)$; the data of the form f determines the form q by the relation $q(u) = f(u,u)$.

The three following relations are commonly used in practice :

1. $q(u+v) = q(u) + q(v) + f(u,v) + f(v,u)$,
2. $q(u-v) = q(u) + q(v) - f(u,v) - f(v,u)$,
3. $q(u+v) - q(u-v) = 2(f(u,v) + f(v,u))$.

The restriction of q to the sub-space $\mathcal{S}_2(E)$ of the symmetric bilinear forms on E induces an isomorphism of $\mathcal{S}_2(E)$ on $\mathcal{Q}(E)$. The inverse image of a quadratic form q by this isomorphism is then called the *polar form* of q. In other words, the symmetric bilinear form defined by the Relation (10.3) is the so-called polar form of q. Hence, we have the two following identities (that can be easily deduced from the previous relations) :

$$\begin{cases} 2f(u,v) &= q(u+v) - q(u) - q(v)\,, \\ 4f(u,v) &= q(u+v) - q(u-v)\,. \end{cases}$$

Let q be a quadratic form of polar form f. We have also the two following inequalities, for each $(u,v) \in E^2$:

[1] Moreover, the application F is symmetric.

- *Cauchy-Schwartz's inequality* :
  if q is positive, then :$f^2(u,v) \le q(u)\,q(v)$, the equality is obtained if u and v are colinear and only when the form is positive definite.
- *Minkowski's inequality* :
  if q is positive, then : $\sqrt{q(u+v)} \le \sqrt{q(u)} + \sqrt{q(v)}$, with equality if $v = 0$ or $\exists \alpha \in \mathbb{R}_+$ such that $x = \alpha v$ and only if the form is positive definite.

10.1.4 Distances and norms

Let us consider now the field of real numbers, $\mathbb{R}$. We call *distance* any application $d : \mathbb{R} \times \mathbb{R} \longmapsto \mathbb{R}_+$ such that $\forall (x,y) \in \mathbb{R}^2$:

$$\begin{array}{rcl} x = y & \Longleftrightarrow & d(x,y) = 0 \\ d(x,y) & = & d(y,x) \\ d(x,y) & \le & d(x,z) + d(z,y)\,. \end{array}$$

On the other hand, we call *norm* on $\mathbb{R}$ any application N from $\mathbb{R}$ to $\mathbb{R}_+$ such that $\forall (x,y) \in \mathbb{R}^2$:

$$\begin{array}{rcl} N(x) = 0 & \Longleftrightarrow & x = 0 \\ N(x+y) & \le & N(x) + N(y) \\ N(\lambda x) & = & \lambda\, N(x)\,. \end{array}$$

Each vector of norm 1 is so-called *unit vector*.

For instance, the three following applications are norms on $\mathbb{R}$:

1. $N_1(x) = \sum_{i=1}^{n} |x_i|$,
2. $N_2(x) = \sqrt{\sum_{i=1}^{n} |x_i|^2}$,
3. $N_\infty(x) = \sup |x_i|$,

and we write $N_i(x) = ||x||$.

We will then show which relation exists between a symmetric positive bilinear form and a norm or a distance.

Norm associated with a quadratic form. Let us consider a positive symmetric bilinear form, non-degenerate, f on $\mathbb{R}$ and let q be the associated quadratic form. We are now trying to establish that the application $\sqrt{q(x)}$ is a norm on $\mathbb{R}$. We have already noticed that (Minkowski's inequality) :

$$\sqrt{q(x+y)} \le \sqrt{q(x)} + \sqrt{q(y)}\,.$$

Moreover, by definition the following relation holds :

$$q(\lambda\, x) = \lambda^2\, q(x)$$

we can then deduce that :

$$\sqrt{q(\lambda\, x)} = |\lambda|\, \sqrt{q(x)}\,.$$

As the form f is non-degenerate, we have a third relation :

$$q(x) = 0 \Longleftrightarrow x = 0\,,$$

and therefore we can deduce that the application $x \longmapsto \sqrt{q(x)}$ is a norm on $\mathbb{R}$.

Dot product. Any non-degenerate symmetric positive bilinear form f on $\mathbb{R}$ can be written in an orthonormal basis :

$$f(x,y) = \sum_{i=1}^{n} x^i\, y^i\,, \qquad \text{or also} \qquad q(x) = f(x,x) = \sum_{i=1}^{n} (x^i)^2\,.$$

Among all the possible forms, we pick one particular form f the so-called *dot product* on $\mathbb{R}$ which is written as :

$$f(x,y) = \langle x, y\rangle\,.$$

Thus introduced, the norm $x \rightarrow \sqrt{\langle x, x\rangle}$ is called the *Euclidean norm.* This obviously means that the associated quadratic form q is the square of the norm :

$$q(x) = \langle x, x\rangle = ||x||^2\,.$$

Distance between two points. Using the Euclidean norm, we can define the distance between two points $(P,Q) \in \mathbb{R}^2$ as follows :

$$d(P,Q) = ||P - Q||\,. \tag{10.4}$$

By extension, we also say that $||x||$ is the *length* of vector x.

We have thus the two following classical results in $\mathbb{R}$:

- *Cauchy-Schwartz's inequality* : $||x + y|| \leq ||x|| + ||y||$ and
- *Pythagorus's theorem* :

$$\left\| \sum_{i=1}^{n} x_i \right\|^2 = \sum_{i=1}^{n} ||x_i||^2\,.$$

10.1.5 Matrix form of a quadratic form

A quadratic form q on E is so-called *positive* (resp. *strictly positive*) and denoted $q \geq 0$ (resp. $q > 0$), if $q(u) \geq 0$ (resp. $q(u) > 0$) for each u (resp. $u \neq 0$) of E.

Hence, whatever the basis $a = (a_1, ..., a_n)$ of E, the determinant of the matrix $\mathcal{M} = [f(a_i, a_j)]$ of q in this basis is positive or null (resp. strictly positive) if $q \geq 0$ (resp. $q > 0$).

A symmetric matrix $\mathcal{M} = [f_{ij}]$ on (the ordered field) K is called *positive* if for any column-vector $U \neq 0$:

$$q(U) =^t U\mathcal{M}U = \sum_{i,j} f_{ij} u_i u_j = \sum_{i} f_{ii} (u_i)^2 + 2 \sum_{i<j} f_{ij} x_i u_j$$

is positive. In other words, the matrix $\mathcal{M}$ is the matrix of a positive quadratic form q on K.

Notice also that the eigenvalues of a symmetric operator are real numbers. Any symmetric operator is diagonalizable in an orthogonal basis. This means that there is always an orthogonal basis of eigenvectors for any symmetric operator. We will see an important application of this property concerning the diagonalization of two quadratic forms.

Having recalled these classical results of linear algebra, it is now possible to introduce the notions of length and metric which are commonly involved in mesh generation and mesh modification algorithms.

10.2 Distances and lengths

We have seen in Section (10.1.3) that the data of a symmetric bilinear form $f(u,v) = \frac{1}{2}(q(u+v) - q(u) - q(v))$ is equivalent to the data of a quadratic form q on $\mathbb{R}$. From the notion of metric space, which allows us to define the distance between two points, we will show how to compute the length of a segment.

10.2.1 Classical length

The notion of length in a metric space is related to the notion of metric and, thus, to a suitable definition of the dot product in the given vector space.

Notion of metric. Assume that at any point P of $\mathbb{R}^d$ a metric tensor is given, as a $(d \times d)$ symmetric definite positive matrix $\mathcal{M}(P)$, (*i.e.*, non-degenerate). For example, in two dimensions, we consider :

$$\mathcal{M}(P) = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \tag{10.5}$$

such that $a > 0$, $c > 0$ and $ac - b^2 > 0$, for $a, b, c \in \mathbb{R}$ (notice that these values depend on P, *i.e.*, $a = a(P)$, etc.). If the field of tensors thus defined is known, it induces a Riemannian structure over $\mathbb{R}^d$.

Remark 10.1 *In the case where $\mathcal{M}(P)$ does not depend on P, the matrix so defined has real coefficients and we again find the classical Euclidean case (where the metric is independent of the position).*

Dot product. The dot product of two vectors in the classical Euclidean space for a given metric $\mathcal{M}(P)$ can be defined as :

$$\langle u, v\rangle_{\mathcal{M}(P)} = u\,\mathcal{M}(P)\,v\,, \tag{10.6}$$

and therefore, considering the Euclidean norm introduced in the previous section, the norm of a vector u is given by the relation :

$$||u|| = \sqrt{\langle u, u\rangle_{\mathcal{M}(P)}} = \sqrt{{}^t u\,\mathcal{M}(P)\,u}\,. \tag{10.7}$$

Notion of length (general case). Having recalled the notions of metric and dot product, we will now introduce the notion of the length of a vector.

In the Euclidean space $\mathbb{R}^2$ or $\mathbb{R}^3$, supplied with the Euclidean norm, we have seen that :

$$f(u, v) = \langle u, v\rangle \quad i.e., \quad q(u) = \langle u, u\rangle = ||u||^2\,,$$

which allows us to see that $\mathcal{M} = I_d$ (as compared with the above definition of the dot product).

This will allow us to compute the length of any vector u, which is indeed the distance between the two endpoints of this vector, using the norm :

$$||u|| = \sqrt{{}^t u\,\mathcal{M}(P)\,u}\,. \tag{10.8}$$

To summarize, given a quadratic form q, computing a length consists in using the dot product associated to q. Hence, the dot product can be formally written as : $\langle .,.\rangle_q$ or also as $\langle .,.\rangle_{\mathcal{M}}$ where $\mathcal{M}$ is the matrix (symmetric positive definite) associated with the quadratic form q.

Notion of length (particular case). The length notion can be also retrieved in differential geometry, as will be seen in the following chapter. Let us consider the space supplied with a Riemannian structure induced by a metric $\mathcal{M}_\gamma$. We consider the curve γ that is the shortest path between two given points A and B. Such a curve is a so-called *geodesic*. Assume a parameterization $\gamma(t)$ of the arc γ of class C^k $(k \geq 1)$ is known, such that $\gamma(0) = A$ and $\gamma(1) = B$. Then, the *length* $L(\gamma)$ of the arc is defined as :

$$L(\gamma) = \int_0^1 ||\gamma'(t)||dt = \int_0^1 \sqrt{{}^t\gamma'(t)\,\mathcal{M}_\gamma\,\gamma'(t)}\,dt\,. \tag{10.9}$$

We then call *distance* between two points, the lower bound of the length of the curves connecting these points. Computing $L(\gamma)$ requires knowing $\gamma(t)$, which turns out to be difficult in practice. That is why we consider the case where the metrics are independent of the position (which reduces the problem to the classical Euclidean case, cf. Remark (10.1)), for which the geodesics are straight (line) segments.

So, the restriction of a paremetrized arc $\gamma(t)$, $t \in [a, b]$ to a vector $\overrightarrow{AB}$ with the parameterization $\gamma(t) = A + t\overrightarrow{AB}$, $t \in [0, 1]$ and $\gamma(0) = A, \gamma(1) = B$ allows us to write the length $L(\gamma)$ of the segment as :

$$L(\gamma) = \int_0^1 \sqrt{{}^t\gamma'(t)\,\mathcal{M}_\gamma\,\gamma'(t)}\,dt \tag{10.10}$$

where M_γ represents the metric specification along γ. Hence, noticing that $\gamma'(t) = \overrightarrow{AB}$, we have :

$$L(\gamma) = \int_0^1 \sqrt{{}^t\overrightarrow{AB}\,\mathcal{M}_\gamma\,\overrightarrow{AB}}dt\,.$$

Writing $\mathcal{M}_\gamma = \mathcal{M}$ (*i.e.*, the metric is independent of the position), we obtain the relation :

$$L(\gamma) = \sqrt{{}^t\overrightarrow{AB}\,\mathcal{M}\,\overrightarrow{AB}}\,.$$

And, in the particular case where $\mathcal{M}_\gamma = I_d$, we have finally :

$$L(\gamma) = \int_0^1 ||\gamma'(t)||dt = \sqrt{\langle\overrightarrow{AB}, \overrightarrow{AB}\rangle} = ||\overrightarrow{AB}||\,,$$

which is obviously the expected result.

10.2.2 Unit length

The key is now to define a way to compute the lengths in the case where various metrics (*i.e.*, different from the above classical Euclidean case) are specified. Thus, we want to change the definition of q (or that of $\mathcal{M}$) resulting in a different expression for $\langle .,.\rangle_q$ (or $\langle .,.\rangle_{\mathcal{M}}$).

Notice firstly that two situations will be of particular interest. The are related to the way q (or $\mathcal{M}$) is defined. The first case corresponds to the Euclidean case where q is defined in a global fashion. On the other hand, cases where q depends on the spatial position (say $q = q_t$ or, similarly, $\mathcal{M} = \mathcal{M}(t)$) lead to a Riemannian context.

Unit length. Let us consider a given basis $a^k, k = 1, d$ of unit vectors in $\mathbb{R}^d$ and d positive real values λ_k. We want to define a metric $\mathcal{M}$ where the norm, denoted by $||.||_{\mathcal{M}}$, is such that satisfying the relation $||u||_{\mathcal{M}} = 1$ means that vector u conforms to the pairs (λ_k, u^k)'s.

To make these notions more precise, we first give some simple examples, then we introduce the general notion of unit length.

In the first example, we want to define segments (vectors) of constant length h, irrespective of the direction. This problem is *isotropic* by nature. Indeed, the geometric locus of all points P distant from h from a given point O is a circle (a sphere), centered at O of radius h.

In practice, we want to define metric $\mathcal{M}$ such that :

$$||\overrightarrow{OP}||_{\mathcal{M}} = 1 \iff ||\overrightarrow{OP}|| = h\,.$$

Notice (cleverly) that the diagonal matrix Λ having all its coefficients λ_k equal to $\frac{1}{h}$ leads to a matrix :

$$\mathcal{M} = \Lambda^2 = \frac{1}{h^2} I_d\,,$$

which is a solution of the previous equation. Indeed, using Relation (10.8), with $u = \overrightarrow{OP}$ and $||\overrightarrow{OP}|| = h$, we obtain :

$$||\overrightarrow{OP}||_{\mathcal{M}} = \sqrt{{}^t\overrightarrow{OP}\,\mathcal{M}\,\overrightarrow{OP}} = \sqrt{{}^t\overrightarrow{OP}\,\Lambda^2\,\overrightarrow{OP}} = \sqrt{{}^t\overrightarrow{OP}\,\frac{I_d}{h^2}\,\overrightarrow{OP}} = \frac{||\overrightarrow{OP}||}{h} = 1\,.$$

In fact, according to Relation (10.2), we can observe that the metric defined in this way corresponds to a circle (a sphere). Let consider the bilinear form

$$f(u,v) = \frac{{}^t uv}{h^2}\,.$$

Then, for example in $\mathbb{R}^2$, we have the relation :

$$f(u,u) = \frac{u_x^2 + u_y^2}{h^2}\,,$$

which, in the case where $f(u,u) = 1$ defines a circle of radius h.

Let us now consider an *anisotropic* example. More precisely, if e_k $(k = 1, d)$ denotes a vector of the canonical basis of $\mathbb{R}^d$, we want to define segments (vectors) of length h_k in the direction e_k.

Let Λ be the $d \times d$ diagonal matrix in which all diagonal coefficients λ_k are set to $\dfrac{1}{h_k}$. Then, we define a transformation $\mathcal{T}$ such that

$$\mathcal{T}(e_k) = e_k\,, \text{for } k = 1,..,d\,.$$

We introduce the matrix $\mathcal{M} = {}^t\mathcal{T}\Lambda\Lambda\mathcal{T}$ and we define

$$||u||_{\mathcal{M}} = \sqrt{{}^t u\,\mathcal{M}\,u}\,.$$

Hence, given a point O, the points P such that $||\overrightarrow{OP}||_{\mathcal{M}} = 1$ is within an ellipse (an ellipsoid) centered at O, aligned with the e_k's, whose radii are the h_k's. Similarly to the isotropic case, we find, for instance in two dimensions :

$$f(u,u) = \frac{u_x^2}{h_x^2} + \frac{u_y^2}{h_y^2}\,,$$

which actually is the expected classical result.

Finally, the last example corresponds to the general anisotropic case. In this case, segments (vectors) of length h_k are desired in the direction a_k. Following the same scheme as in the previous example, the transformation $\mathcal{T}$ is now such that :

$$\mathcal{T}(a_k) = e_k\,, k = 1,..,d\,,$$

and similarly,

$$\mathcal{T}^{-1}(e_k) = a_k\,, k = 1,..,d\,.$$

Then, the relation $||\overrightarrow{OP}||_{\mathcal{M}} = 1$ defines an ellipse (an ellipsoid) centered at O, aligned with the a_k and such that the radii are the given h_k's.

A simple way of proving this is to associate a point P' with each point P using the relation $OP = \mathcal{T}^{-1}OP'$. Then, $||\overrightarrow{OP}||_{\mathcal{M}} = 1$ leads to writing the relations :

$$1 = \sqrt{{}^t\overrightarrow{OP}\mathcal{M}\overrightarrow{OP}} = \sqrt{{}^t\overrightarrow{OP}\,{}^t\mathcal{T}\Lambda^2\mathcal{T}\overrightarrow{OP}} = \sqrt{{}^t\overrightarrow{OP'}\,{}^t\mathcal{T}^{-1}\,{}^t\mathcal{T}\Lambda^2\mathcal{T}\mathcal{T}^{-1}\,\overrightarrow{OP'}}\,,$$

which can be reduced to the relation :

$$1 = \sqrt{{}^t\overrightarrow{OP'}\Lambda^2\,\overrightarrow{OP'}}\,,$$

which is indeed the relation showing that the point P' belongs to an ellipse (in two dimensions) centered at O, aligned with the e_k and whose radii are the h_k. Then, using the relation relating points P and P', it is easy to see that P belongs (in two dimensions) to the ellipse with the same center and the same radii, but now aligned with the a^k.

Remark 10.2 *It could be noticed that the above general form can be reduced to the first two cases, provided a suitable choice of Λ and of $\mathcal{T}$. So, in the first example, $\mathcal{T}$ is the identity matrix I_d and $\Lambda^2 = \frac{I_d}{h^2}$. In the second example, we have again $\mathcal{T} = I_d$ while Λ is the diagonal matrix whose coefficients are h_k^{-1}.*

A global definition. In this case, the metric (*i.e.*, q or $\mathcal{M}$) is globally defined, thus meaning that the notion of unit length is the same at any point location. Computing a length is then easy since the matrix involved in such a calculation is a constant one.

A local definition. In this case, the metric varies from point to point and the notion of unit length is different according to the spatial position. Actually, if we consider the matrix $\mathcal{M}$, this matrix is a function of the current position and thus can be expressed as $\mathcal{M}(t)$, where t is a parameter value. Unlike the previous case, the calculation of a length is more tedious. In practice, the matrix involved in the formula is no longer constant, thus leading us to consider approximate solutions for the length calculation.

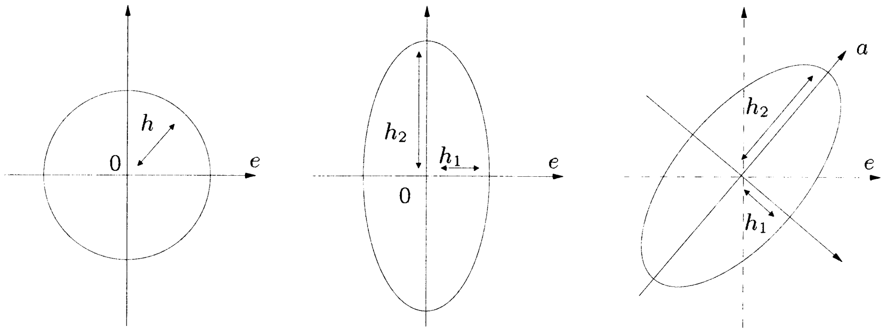

Figure 10.1: *The geometric interpretation of the various metrics. An isotropic metric leads to a circle (left-hand side), an anisotropic metric leads to an ellipse (aligned with the canonical basis, middle, or aligned with any arbitrary orthogonal vectors, right-hand side).*

10.2.3 Applications

Let us consider a given specification. This can be expressed in terms of sizes or in terms of directional features and related sizes. Then, the previous material enables us to characterize this specification as a unit length defined in a suitable space.

Actually, two categories of metric specifications can be exhibited, each of which includes two classes of definitions. Independently of this classification, the metrics are either isotropic or anisotropic by nature.

We have already mentioned that the case where the metric is constant over the space is equivalent to the classical Euclidean situation. When the metric varies from point to point (*i.e.*, is not constant from one point to another), the context of the study is then Riemannian (actually, the field of tensors induces a Riemannian structure over $\mathbb{R}^d$). The length calculation then requires a special effort.

10.3 Metric-based operations

In this section, we will discuss various metric-based manipulation methods. Indeed, in many applications, several metrics can be defined at any point location (of the computational domain). We will then consider different ways of going back to the specification of a unique metric. Actually, the metric-based operations are considered from the point of view of operations dealing with the associated quadratic forms.

10.3.1 Simultaneous reduction of two metrics

Given two metrics $\mathcal{M}_1$ and $\mathcal{M}_2$ (or similarly two quadratic forms q_1 and q_2), the problem here is to express these two metrics in a basis where the associated matrices are both diagonal. In general, no such basis e exists that is orthogonal for both q_1 and q_2. Such a basis is one of eigenvectors for the operator $f_1^{-1} f_2$ (f_1 and f_2 being the linear applications associated with q_1 and q_2), hence a basis allowing us to diagonalize this operator.

To this end, we can discuss either from the quadratic forms or from the matrices related to the given metrics. For the sake of convenience, we will follow the second approach.

Let us denote by $\mathcal{M}_1$ and $\mathcal{M}_2$ the two $d \times d$ matrices related to the two given quadratic forms. The simultaneous reduction of two positive quadratic forms is possible as soon as one of them is defined, which means that the associated matrix is invertible. Assume then that $\mathcal{M}_1$ is positive definite[2].

To obtain a basis where both $\mathcal{M}_1$ and $\mathcal{M}_2$ are diagonal, we introduce the matrix $\mathcal{N}$ defined as :

$$\mathcal{N} = \mathcal{M}_1^{-1} \mathcal{M}_2 .$$

The matrix $\mathcal{N}$ being $\mathcal{M}_1$-symmetric, it can thus be diagonalized. Let e_1 and e_2 be the two eigenvectors of $\mathcal{N}$. These vectors define a basis of $\mathbb{R}^d$ and we can write :

$${}^t e_2 \, \mathcal{M}_1 \, e_1 = {}^t e_2 \, \mathcal{M}_2 \, e_1 = 0 ,$$

thus meaning that e_1 and e_2 are $\mathcal{M}_i$-orthogonal ($i = 1, 2$). To establish this property, we consider λ_1 and λ_2 the two eigenvectors associated with the two previously defined eigenvectors. Then, we have the relations :

$$\mathcal{N} e_1 = \lambda_1 e_1 \quad \text{as well as} \quad \mathcal{N} e_2 = \lambda_2 e_2 \quad i.e.,$$

$$\mathcal{M}_1^{-1} \mathcal{M}_2 \, e_1 = \lambda_1 e_1 \quad \text{and} \quad \mathcal{M}_1^{-1} \mathcal{M}_2 \, e_2 = \lambda_2 e_2 .$$

By applying $\mathcal{M}_1$ to the left, the previous relations become :

$$\mathcal{M}_2 \, e_1 = \lambda_1 \mathcal{M}_1 \, e_1 \quad \text{and} \quad \mathcal{M}_2 \, e_2 = \lambda_2 \mathcal{M}_1 \, e_2 . \qquad (10.11)$$

From the first relation we deduce :

$${}^t e_2 \mathcal{M}_2 \, e_1 = \lambda_1 \, {}^t e_2 \mathcal{M}_1 \, e_1 ,$$

and from the second one :

$${}^t e_1 \mathcal{M}_2 \, e_2 = \lambda_2 \, {}^t e_1 \mathcal{M}_1 \, e_2 ,$$

whose transposed is :

$${}^t e_2 \mathcal{M}_2 \, e_1 = \lambda_2 \, {}^t e_2 \mathcal{M}_1 \, e_1 ,$$

then, by identification, we obtain :

$$\lambda_1 \, {}^t e_2 \mathcal{M}_1 \, e_1 = \lambda_2 \, {}^t e_2 \mathcal{M}_1 \, e_1 ,$$

which finally implies that :

$${}^t e_2 \mathcal{M}_1 \, e_1 = {}^t e_2 \mathcal{M}_2 \, e_1 = 0 .$$

[2] Trivially, we will not consider the case where the two matrices are linked by a relation like $\mathcal{M}_1 = \alpha \mathcal{M}_2$.

Moreover, the Relation (10.11) leads to :

$$^te_1\mathcal{M}_2\,e_1 = \lambda_1\,^te_1\mathcal{M}_1\,e_1 \quad \text{and} \quad ^te_2\mathcal{M}_2\,e_2 = \lambda_2\,e_2\mathcal{M}_1\,e_2\,,$$

which can also be written as follows :

$$q_2(e_1) = \lambda_1 q_1(e_1) \quad \text{and} \quad q_2(e_2) = \lambda_2 q_1(e_2)\,,$$

by using the two corresponding quadratic forms.

Expression of a vector in the eigenbasis. In the basis defined by $[e_i, e_2]$, any vector v can be written as $v = x_1e_1 + x_2e_2$. The two quadratic forms $q_1(v)$ and $q_2(v)$ are represented by two diagonal matrices. This result is left as an exercise :

Exercise 10.1 *Define $\alpha_{i,j}$ the coefficients such that :*

$$^tv\mathcal{M}_1v = \sum_{i,j}\alpha_{i,j}x_ix_j\,.$$

Prove that $\alpha_{i,j} = 0$ for $i \neq j$ and show that $\alpha_{i,i} = {}^te_i\mathcal{M}_1\,e_i$. Similarly, reconsider the same exercise with the matrix $\mathcal{M}_2$.

Expression of the matrices in the eigenbasis. Let $\mathcal{M}_1^e$ and $\mathcal{M}_2^e$ be the matrices obtained by replacing $\mathcal{M}_1$ and $\mathcal{M}_2$ by the corresponding forms in the eigenbasis.

Exercise 10.2 *Express the transformation defined by the matrix $\mathcal{R} = (e_1, e_2)$, find the expression of $\mathcal{M}_1^e = {}^t\mathcal{R}\mathcal{M}_1\mathcal{R}$ and verify that a diagonal matrix results from this operation. Similarly, consider the coefficients of the matrix $\mathcal{M}_2^e$.*

Remark 10.3 *Notice that $\mathcal{M}_1^e$ is I_d, the identity matrix, if we normalize e_1 and e_2 with respect to $\mathcal{M}_1$. In this case we have, for the corresponding quadratic forms :*

$$q_1(e_1) = q_1(e_2) = 1 \quad \textit{as well as} \quad q_2(e_1) = \lambda_1 \quad \textit{and} \quad q_2(e_2) = \lambda_2\,.$$

10.3.2 Metric interpolation

For the sake of clarity, we only consider here the two dimensional case. Let P_1 and P_2 be two points in $\mathbb{R}^2$ and let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two metrics associated with these points. The problem here is to find a metric $\mathcal{M}(t)$ defined on the segment $[P_1, P_2] = [P_1 + t\,(P_2 - P_1)]$ for each t in $[0,1]$. Moreover, the desired metric must be such that :

$$\mathcal{M}(0) = \mathcal{M}_1 \quad \text{and} \quad \mathcal{M}(1) = \mathcal{M}_2\,,$$

and must vary monotonously between these two values.

Achieving such a metric is actually equivalent to performing a *metric interpolation*. To this end, various techniques can be considered.

An intuitive method. To give the idea of this kind of method, we consider the isotropic situation. Then, the desired solution can be obtained trivially. Indeed, if the metrics are simply λI_d and μI_d, then the expected sizes are respectively $h(0) = 1/\sqrt{\lambda}$ for $\mathcal{M}_1$ (at point P_1) and $h(1) = 1/\sqrt{\mu}$ for $\mathcal{M}_2$ (at point P_2). Hence, assuming that an arithmetic (linear) size distribution is specified, the interpolation function is defined as follows :

$$\mathcal{M}(t) = \frac{1}{(h(0) + t(h(1) - h(0)))^2}\,I_d\,, \qquad 0 \leq t \leq 1\,, \tag{10.12}$$

with $\mathcal{M}(0) = \mathcal{M}_1$ and $\mathcal{M}(1) = \mathcal{M}_2$.

Notice that other types of distributions can be considered, for instance a geometric distribution (see below).

In the anisotropic case, several approaches can be considered. By analogy with the isotropic case where the metric is usually written as $\mathcal{M} = \frac{1}{h^2}\,I_d$, we observe that the variation related to the h's is "equivalent" to the variation related to the $\mathcal{M}^{-1/2}$'s. Hence, we obtain the following interpolation scheme :

$$\mathcal{M}(t) = \left((1-t)\mathcal{M}_1^{-1/2} + t\mathcal{M}_2^{-1/2}\right)^{-2}\,, \qquad 0 \leq t \leq 1\,. \tag{10.13}$$

Computing $\mathcal{M}^{-1/2}$ requires evaluating the eigenvalues of $\mathcal{M}$, which is tedious. To avoid this problem, we can consider the interpolation as :

$$\mathcal{M}(t) = ((1-t)\mathcal{M}_1^{-1} + t\mathcal{M}_2^{-1})^{-1}\,, \qquad 0 \leq t \leq 1\,, \tag{10.14}$$

and notice that this relation emphasizes the smallest sizes (*i.e.*, the weakest values of h).

The interpolation scheme based on a metric exponent is properly defined. Actually,

- if $\mathcal{M}$ is a metric, then $t\mathcal{M}^\alpha$ is also a metric, when $t > 0$ and α are two arbitrary real values;
- if $\mathcal{M}_1$ and $\mathcal{M}_2$ are two metrics, $\mathcal{M}_1 + \mathcal{M}_2$ is also a metric.

Proving these results requires only making sure that, in each case, the resulting matrices are symmetric and positive definite.

Notice however, that this kind of intuitive interpolation presents some weaknesses. In particular, the variations in terms of h cannot be explicitly controlled. Thus, in the following, we consider the *simultaneous reduction* of two metrics. Conversely (see below) the intuitive method gives a solution when the interpolation is performed on a triangle and not only along a line.

A method based on the simultaneous reduction. The interpolation metric is obtained after a two-step algorithm :

Step 1 : using the above simultaneous reduction, we write both $\mathcal{M}_1$ and $\mathcal{M}_2$ in a diagonal form, Step 2 : according to the interpolation between P_1 and P_2, we complete the desired interpolation between the metrics.

Thus, let e_1 and e_2 be the two eigenvectors of $\mathcal{N} = \mathcal{M}_1^{-1}\mathcal{M}_2$, the eigenvalues of the metric $\mathcal{M}_1$ are the λ_i's such that $(\lambda_i = {}^t e_i \mathcal{M}_1 e_i)_{i=1,2}$ and that of the metric $\mathcal{M}_2$, the μ_i's such that $(\mu_i = {}^t e_i \mathcal{M}_2 e_i)_{i=1,2}$. Any vector $X = x_1 e_1 + x_2 e_2$ in $\mathbb{R}^2$, written in the basis $[e_1, e_2]$, is such that :

$$ {}^t X \mathcal{M}_1 X = \lambda_1 x_1^2 + \lambda_2 x_2^2 \quad \text{and} \quad {}^t X \mathcal{M}_2 X = \mu_1 x_1^2 + \mu_2 x_2^2 . $$

Now, we define $(h_{1,i} = \dfrac{1}{\sqrt{\lambda_i}})_{i=1,2}$ and $(h_{2,i} = \dfrac{1}{\sqrt{\mu_i}})_{i=1,2}$. Then, the value $h_{1,i}$ (resp. $h_{2,i}$) is the unit length in the metric $\mathcal{M}_1$ (resp. $\mathcal{M}_2$) along the axis e_i. And the interpolation metric between $\mathcal{M}_1$ and $\mathcal{M}_2$ is defined using the formula :

$$ \mathcal{M}(t) = {}^t\mathcal{P}^{-1} \begin{pmatrix} \dfrac{1}{H_1^2(t)} & 0 \\ 0 & \dfrac{1}{H_2^2(t)} \end{pmatrix} \mathcal{P}^{-1} \qquad t \in [0,1], $$

where $\mathcal{P}$ is the matrix formed by the column-vector (e_1, e_2), and $(H_1(t), H_2(t))$ are two monotonous continuous functions such that $H_i(0) = h_{1,i}$ and $H_i(1) = h_{2,i}$ for $i = 1, 2$. To complete the definition of this interpolation, we have still to express the terms $H_i(t)$.

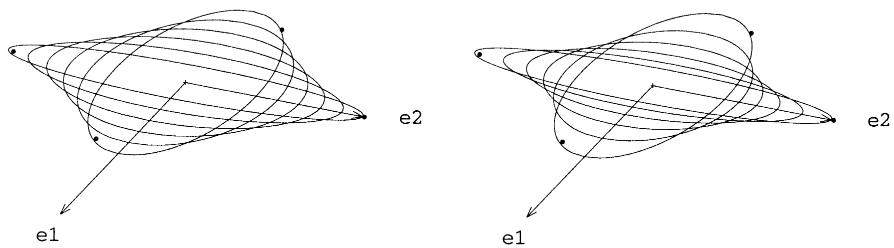

Figure 10.2: *Metric interpolation : continuous variation of the metric between $\mathcal{M}_1$ and $\mathcal{M}_2$. left-hand side, linear interpolation, right-hand side, geometric interpolation.*

Depending on the expected result, various choices can be made. In practice, we can consider the following interpolation functions :

- a linear function : $H_i(t) = h_{1,i} + t\,(h_{2,i} - h_{1,i})$,
- a geometric function : $H_i(t) = h_{1,i} \left(\dfrac{h_{2,i}}{h_{1,i}}\right)^t$,
- a sinusoïdal function : $H_i(t) = \frac{1}{2}\left(h_{1,i} + h_{2,i} + (h_{1,i} - h_{2,i})\cos(\pi t)\right)$.

Notice that this interpolation is only controlled along the directions of the axes e_1 and e_2. As an example, we present Figure 10.2 which illustrates the two initial metrics (represented with small dots) and the interpolated metrics in the case of a linear function (left-hand side) and of a geometric function (right-hand side).

Remark also that the previous discussion, in two dimensions, extends to three dimensions.

Remark 10.4 *The metric interpolation method by means of simultaneous reduction is unlikely to be suitable when looking for the solution inside a triangle. Indeed, it is well suited to "edge"-type interpolation, i.e., the metrics at the edge endpoints are given and the metric at any point of this edge is sought. For a triangle interpolation, more intuitive methods give reasonable solutions.*

10.3.3 Metric intersection

Now we face a different kind of problem. Given a point P, we assume that several metrics $\mathcal{M}_i$ are supplied at this point. The problem here is to find a unique metric $\mathcal{M}$ that somehow reflects, in a sense that we will specify, the nature of the initial metrics.

For the sake of simplicity, we consider only the two-dimensional case, while noting however that the relations that will be established also apply in three dimensions (replacing a circle by a sphere, an ellipse by an ellipsoid).

First, we discuss the case where two metrics are supplied, and we consider the unit circles (in fact, ellipses) associated with the two original metrics. The desired solution is then a metric associated with the intersection of these two ellipses. As in general, the result is not an ellipse, we can consider one of the ellipses that fits in this intersection area. In this way, we define a so-called *intersection metric*. According to the choice of the ellipse contained in this intersection region, different solutions can be obtained. One solution consists of considering the largest ellipse, while another attempts to preserve some features (for instance directional) of one of the two initial ellipses. This leads to two solutions which are discussed below.

Metric intersection using the simultaneous reduction scheme. The simultaneous reduction of the two quadratic forms corresponding to two metrics leads to defining the intersection metric related to the two initial metrics as explained in the previous section. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two metrics, the two corresponding unit circles can be expressed in the base associated with the simultaneous reduction of the matrices $\mathcal{M}_1$ and $\mathcal{M}_2$:

$$ {}^t X \mathcal{M}_1 X = \lambda_1 x^2 + \lambda_2 y^2 = 1 \quad \text{and} \quad {}^t X \mathcal{M}_2 X = \mu_1 x^2 + \mu_2 y^2 = 1 \, ; \tag{10.15} $$

the intersection metric $(\mathcal{M}_1 \cap \mathcal{M}_2)$ is then defined as :

$$ (\mathcal{M}_1 \cap \mathcal{M}_2) = {}^t\mathcal{P}^{-1} \begin{pmatrix} \max(\lambda_1, \mu_1) & 0 \\ 0 & \max(\lambda_2, \mu_2) \end{pmatrix} \mathcal{P}^{-1} \tag{10.16} $$

where $\mathcal{P}$ is the matrix mapping the canonical basis to that associated with the simultaneous reduction of the two metrics. Figure 10.3 (left-hand side) depicts the metric intersection of two given metrics based on the simultaneous reduction.

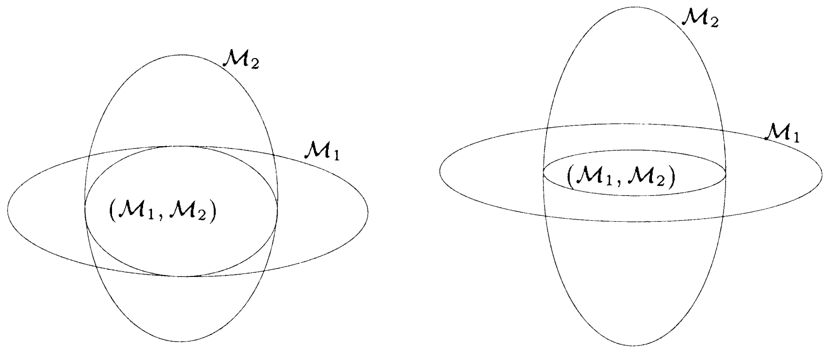

Figure 10.3: *Intersection of two metrics $\mathcal{M}_1 \cap \mathcal{M}_2$ based on the simultaneous reduction of the metrics (left-hand side) and preserving the directions of the metric $\mathcal{M}_1$ (right-hand side).*

When several metrics $(\mathcal{M}_i)_{1 \leq i \leq q}$ are specified at a given point, the resulting intersection metric can be defined using the following formula :

$$(\mathcal{M}_1 \cap \cdots \cap \mathcal{M}_q) = ((...((\mathcal{M}_1 \cap \mathcal{M}_2) \cap \mathcal{M}_3) \cap ...) \cap \mathcal{M}_q)\,. \qquad (10.17)$$

Exercise 10.3 *Prove that $(\mathcal{M}_1 \cap \mathcal{M}_2)$ defines a metric (Hint : check that the relevant properties hold).*

Exercise 10.4 *Is the intersection scheme an associative or commutative scheme ?*

Metric intersection preserving specific directions. The previously described method consists in finding the maximal ellipse included in the intersection region of the initial ellipses. Hence, this requirement does not preserve, in any way, the directions of one or the other of the given metrics. As this latter property can be of great interest[3], we are suggesting a different method, that leads to construct a metric having its directions identical to those of one of the initial metrics. Then, a maximal ellipse with particular directions will be found. In the case depicted in Figure 10.3 (right-hand side), the directions specified in metric $\mathcal{M}_1$ are preferred. The intersection metric $(\mathcal{M}_1 \cap \mathcal{M}_2)$ is defined by

$$(\mathcal{M}_1 \cap \mathcal{M}_2) = \omega\, \mathcal{M}_1 \quad \text{with} \quad \omega = \max\left(\frac{\mu_1}{\lambda_1}, \frac{\mu_2}{\lambda_2}, 1\right) \qquad (10.18)$$

if one wants to preserve the shape of the metric $\mathcal{M}_1$ (μ_i and λ_i also denoting the eigenvalues of the matrices).

10.3.4 Metric smoothing

Given a metric and irrespective of its nature (*i.e.*, related to a geometry or resulting from the physics of the problem considered), there is no guarantee that a mesh

[3] For instance, when triangulating some surfaces.

strictly based on this metric will conform to the whole set of requirements. Several undesirable features can be encountered. It seems indeed obvious that in two dimensions, it is not possible to obtain a mesh composed of equilateral triangles if the given metric presents great size variations.

Variation and shock of a metric. In the isotropic case, the metric at an arbitrary point P can be written, as seen before, as :

$$\mathcal{M}(P) = \frac{1}{h(P)^2} I_d\,,$$

where I_d is the unit matrix of dimension d and $h(P)$ is the desired size at P. Hence, for an edge AB where we want to have $h(A)$ at A, $h(B)$ at B, the length of the segment is :

$$l(AB) = ||\overrightarrow{AB}|| \int_0^1 \frac{1}{h(t)}\, dt$$

where $h(t)$ represents a continuous interpolation function defined on $[0,1]$ such that $h(0) = h(A)$ and $h(1) = h(B)$. This is equivalent to parameterizing the edge AB by $(1-t)\,A + t\,B$ and to denoting in a similar fashion $h(t)$ and $h(P)$, the value of h at the current point P parameterized by t.

Remark 10.5 *The function h being chosen, the previous expression allows us to go from discrete data (in A and B only) to continuous data (along the whole segment AB).*

As mentioned, $h(A)$ and $h(B)$ can be more or less "compatible" with the Euclidean length of AB. To be able to evaluate this notion numerically, we introduce the following definitions.

Definition 10.6 *The* h-variation, *denoted v_h, related to an edge AB is defined as :*

$$v_h(AB) = \frac{|h(B) - h(A)|}{||\overrightarrow{AB}||}\,.$$

The h-shock, *denoted χ_h, related to AB is defined as :*

$$\chi_h = \max\left(\frac{h(B)}{h(A)}, \frac{h(A)}{h(B)}\right)^{\frac{1}{l(AB)}}\,.$$

In other words, the h-variation v_h along AB, when B tends towards A, represents an approximation of the gradient of the function h. The h-shock measures the distortion of h along AB.

For a mesh, these values are defined at each vertex.

Definition 10.7 *Let A be a given mesh vertex and let P_i be the endpoints of the edges incident to A, not equal to A. We set :*

$$v_h(A) = \max_{P_i} v_h(AP_i) \quad \text{and} \quad \chi_h(A) = \max_{P_i} \chi_h(AP_i)\,.$$

This discrete definition enables us to characterize a mesh, according to a given metric field. The h-variation and h-shock values are then defined as the extrema of the values related to these quantities at the mesh vertices.

In the anisotropic case, we define the same notions based on the direction of the given edge. This is equivalent to finding the intersection of the metric in A (resp. in B) with the edge AB and then using the same scheme with $h(A)$ (resp. $h(B)$ defined by $h(A) = ||\overrightarrow{AA_1}||$ (resp. $h(B) = ||\overrightarrow{BB_1}||$) where A_1 (resp. B_1) is the intersection point of AB with the circle related to the (anisotropic) metric $\mathcal{M}(A)$ (resp. the circle of $\mathcal{M}(B)$).

Notice however, that in the anisotropic case, the resulting Riemannian structured is not able to constrain a size variation in each direction.

Metric smoothing using a correction scheme. Given a mesh and a field of metrics defined, in a discrete fashion, at each mesh vertex, the smoothing procedure aims at constructing a (new) field satisfying a given regularity specified *a priori*, whenever the initial field is not compliant. This is especially the case when the size variation is too great or discontinuous.

The new metric is used to reconstruct a new mesh of the domain[4], that is more adapted to the given specifications. In particular, the quality of the resulting mesh is improved, the created elements being more regular (equilateral triangles, for instance).

Let us consider the isotropic case. The problem consists here in bounding the h-variation v_h of an edge AB by a given threshold ε, $v_h < \varepsilon$, by changing the size specifications $h(A)$ and/or $h(B)$. The new specifications are then determined using the following formulas :

$$\begin{aligned} h(A) &= \min(h(A), h(B) + \varepsilon||\overrightarrow{AB}||) \\ \text{and} \quad h(B) &= \min(h(B), h(A) + \varepsilon||\overrightarrow{AB}||)\,. \end{aligned}$$

Notice that only one of the size specifications is affected (the largest one).

The procedure is quite similar in the anisotropic case. To this end, one has simply to extend the operator min, related to the sizes, to an operator related to metrics. This operator is, as expected, the metric intersection operator previously described. Notice that in this case, the correction applied to an edge does not account for the metric interpolation along the edge. Moreover, this operation may affect the shape of the corresponding ellipses (ellipsoids).

This procedure, based on the notion of h-shock, in the isotropic and anisotropic cases can be found in [Borouchaki *et al.* 1998].

Examples of metric smoothing. Figure 10.4 illustrates the effects of the metric smoothing and correction procedures on a surface mesh. Figure 10.5 represents a prediction of an unstationary transonic flow around a wing profile. The flow parameters are $Re = 10^7$, $Mach = 0.775$ and the angle of incidence $\alpha = 4°$.

[4] The given mesh is seen here as a background mesh and forms, along with the smoothed metric, a control space (Chapter 1).

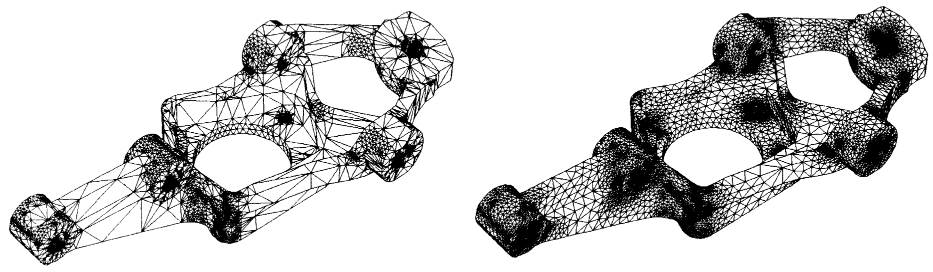

Figure 10.4: *Example of metric correction related to a surface mesh. Left, geometric mesh without correction (data courtesy of the Mac Neal-Schwendler Corp.). Right, geometric mesh after a metric correction by a given value $\varepsilon = 1.5$.*

In these examples, the influence of the metric correction procedure is clearly visible. This procedure is even more important in the numerical computations, as the difficulties usually encountered are related to the possible lack of information during the interpolation of the solutions (in an adaptation scheme) and to the capture of the critical regions.

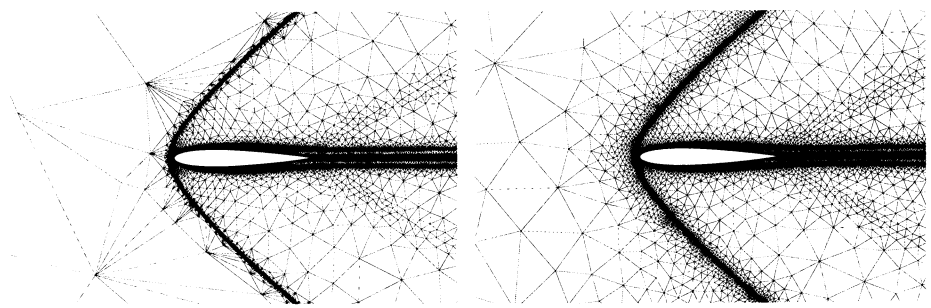

Figure 10.5: *Example of a metric correction on an adapted mesh in a computational fluid simulation. Left, the mesh is adapted without correction. Right, the mesh has been adapted with a metric correction by a given value $\varepsilon = 2$.*

10.4 Metric construction

In the previous sections, we have largely discussed metrics and related operations, on the assumption that these latter were supplied. Now, we give some details on how to define such metrics, especially for surface meshes and regarding a computational scheme based on the finite element method.

10.4.1 Parametric surface meshing

Let Σ be a surface, let σ be its parameterization and let Ω be its parametric space. We want to obtain a surface mesh conforming to some given specifications (in particular, related to the element sizes as well as to the intrinsic properties of Σ, *i.e.*, to conform to the geometry), from a mesh of Ω. In other words, the goal is to control the mesh of Σ by controlling the mesh of Ω. Chapter 15 will deal with this approach more thoroughly.

Assume now that a metric is given on the surface. Let $\mathcal{M}_3$ be the current associated matrix, of dimensions 3×3. The problem is to find the relationship between a length on Σ and the corresponding length in Ω. Thus defined, the problem can be reduced to that of finding the matrix $\mathcal{M}_2$, a 2×2 matrix, related to $\mathcal{M}_3$. To this end, we use the metric induction.

Metric induction. Given a point $X \in \Omega$, the matrix $\mathcal{M}_2(X)$ is the *metric induced* by $\mathcal{M}_3(P)$, $P \in \Sigma$ on the tangent plane to the surface at P. By denoting $\Pi(P)$ the transition matrix from the canonical basis of $\mathbb{R}^3$ to the local basis at the current point P, the desired metric $\mathcal{M}_2(X)$ is defined by the matrix :

$$\mathcal{M}_2 = \lfloor {}^t\Pi \mathcal{M}_3 \Pi \rfloor_2 \,,$$

where the symbol $\lfloor \rfloor_2$ means that we consider the first two columns and the first two lines of the matrix ${}^t\Pi\mathcal{M}_3\Pi$. Given a matrix $\mathcal{M}_3$ we then find by induction a matrix $\mathcal{M}_2$ that enables the lengths on the surface to be controlled via a control of the edge lengths in Ω.

Choice of the surface metrics. The control of the gap between an edge and the surface is obtained using the metric $\mathcal{M}_3$, which has yet to be explained. Hence, to govern the mesh of Σ according to $\mathcal{M}_3$, consists in governing the mesh of Ω with respect to an induced metric $\mathcal{M}_2$. As will be seen in Chapter 15, a judicious choice of $\mathcal{M}_3$ makes it possible to bound the gap between any edge and the surface by a given threshold value ε. A matrix of the form :

$$\mathcal{M}_3(P)_{\rho_1,\rho_2} = {}^t\mathcal{D}(P) \begin{pmatrix} \dfrac{1}{\alpha^2\,\rho_1^2(P)} & 0 & 0 \\ 0 & \dfrac{1}{\beta^2\,\rho_2^2(P)} & 0 \\ 0 & 0 & \lambda \end{pmatrix} \mathcal{D}(P)\,, \tag{10.19}$$

where $\mathcal{D}(P)$ corresponds to the principal directions at P, α and β are suitable coefficients and $\lambda \in \mathbb{R}$, enables us to have an *anisotropic* control of the gap between the geometric, which accounts for the two principal radii of curvature ρ_1 and ρ_2 and for the two principal directions.

We can also consider the case $\rho = \min(\rho_1, \rho_2)$ which leads to a so-called metric

of the minima radius of curvature for which the matrix $\mathcal{M}_3(P)$ can be expressed as :

$$\mathcal{M}_3(P)_\rho = \begin{pmatrix} \dfrac{1}{h^2(P)} & 0 & 0 \\ 0 & \dfrac{1}{h^2(P)} & 0 \\ 0 & 0 & \dfrac{1}{h^2(P)} \end{pmatrix}, \tag{10.20}$$

where the variable $h(P) = \alpha\rho(P)$ is related to the position and α is a suitable coefficient, related to the geometric approximation (*i.e.*, to the gap between the edges of the discretization and the surface).

10.4.2 Finite element simulation with error control

The previously defined fields of metrics are related to the intrinsic properties of the surface, hence to its geometry. We will now focus on metric fields met during the numerical calculations based on finite element methods, in particular in the case of mesh adaptation (Chapters 21 and 22).

Construction of a computational metric. The goal is to construct a metric that allows an homogeneous distribution of the error related to the interpolation of the solutions. The error estimation[5] is analyzed by studying the behavior of $||u - \Pi_h u||$, $\Pi_h u$ being the solution of the discrete model and $||\ ||$ being a suitable norm.

Let us assume that a solution $\Pi_h u$ has already been computed on a former mesh. If the interpolation scheme is linear and piecewise continuous (P^1-type), then the interpolation error can be related to the variations of the variables of the problem and, in particular, to their successive derivatives (gradient, Hessian, etc.).

To show this result, we first focus on a one-dimensional problem with only one unknown u.

Interpolation error for a one-dimensional P^1 problem. Let us consider the segment $[\Pi_h(a), \Pi_h(b)]$, which is the function Π_h that is the linear approximation of the function u between a and b (between $u(a)$ and $u(b)$, in fact).

In order to evaluate the interpolation error between u and $\Pi_h u$, we will perform a local analysis. In other words, we assume that the computed solution is quite close to the real solution. This evaluation is based on various Taylor's formulas. One such formula, when applied to a regular enough function $f(x)$ from $I = [a, b]$ to $\mathbb{R}$, can be written as :

$$f(a) = f(x) = h\,f'(x) + \frac{h^2}{2} f''(x) + \mathcal{O}(h^3)\,, \tag{10.21}$$

[5] as well as the convergence of the numerical approximation.

with $x = a + h$. This formula and, in general, formulas of the same type are actually not well suited for our purpose since h is not necessarily small. Thus, we prefer a formula like :

$$f(a) = f(x) = (a-x)\, f'(x) + \frac{h^2}{2}\, f''(x + t(a-x))\,, \tag{10.22}$$

where t, in $[0,1]$, is a function of both x and a. For our purposes, we fix the function f to be the function that, for $x \in I$ associates $(u - \Pi_h u)(x) = u(x) - \Pi_h u(x)$, assuming thus that $\Pi_h u(a) = u(a)$ and $\Pi_h u(b) = u(b)$. From the previous discussion, regarding u and $\Pi_h u$, we have :

$$u(a) - \Pi_h u(a) = u(x) - \Pi_h u(x) + (a-x)\,(u'(x) - \Pi_h u'(x)) + \frac{(a-x)^2}{2}\, u''(x + t_1(a-x))\,,$$

where now t_1 is a function of a and x. As $u(a) - \Pi_h u(a) = 0$ and as we look for an extremum, x, where $u'(x) - \Pi_h u'(x) = 0$, then we have :

$$0 = (u - \Pi_h u)(x) + \frac{(a-x)^2}{2} u''(x + t_1(a-x))\,.$$

Then, we write the same, based on b. For the above x, we have :

$$0 = (u - \Pi_h u)(x) + \frac{(b-x)^2}{2} u''(x + t_2(b-x))\,.$$

Adding these two relations gives :

$$0 = 2\,(u - \Pi_h u)(x) + \frac{(a-x)^2}{2} u''(x + t_1(a-x)) + \frac{(b-x)^2}{2} u''(x + t_2(b-x))\,.$$

Let M be a majorant of u'' in I, then :

$$|(u - \Pi_h u)(x)| \le \frac{1}{2}\,(\frac{(a-x)^2}{2} + \frac{(b-x)^2}{2})M\,.$$

Then :

$$|(u - \Pi_h u)(x)| \le \frac{1}{2}\, max_{x \in I}(\frac{(a-x)^2}{2} + \frac{(b-x)^2}{2})\, M\,.$$

And finally, $\forall x \in I$, :

$$|u(x) - \Pi_h u(x)| \le \frac{(b-a)^2}{8} M\,. \tag{10.23}$$

The goal is then to see what is the value of $|u(x) - \Pi_h u(x)|$, which means finding the value of $\frac{(b-a)^2}{8} M$ and then to compare this value with a given value ε representing the maximum allowed gap between the function u and the linear approximation $\Pi_h u$.

From a geometric point of view, if x is a point of $[a,b]$, the point $(x, \Pi_h u(x))$ describes the segment $[u(a), u(b)]$ while the point $(x, u(x))$ describes the (unknown) "curve" u. As u is assumed to be sufficiently regular along $[a,b]$, we will then replace the curve by a parabola. Thus, a method allowing us to reach the expected result consists in :

Figure 10.6: *Piecewise linear interpolation in one dimension. Left, are shown the segment $I = [a,b]$, $u(a)$ and $u(b)$, the function $\Pi_h u$, the segment $[u(a), u(b)]$ and the presumed function u. Right, are shown the segment ab and its neighbors.*

- constructing a parabola going through the point $\Pi_h u(a) = u(a)$ and through the point $\Pi_h u(b) = u(b)$ and
- evaluating u'' on I, based on the parabola.

This enables us to find the desired value. If the latter is of the desired order, the meshing step is correct according to point a. If for the given ε, we find a larger or smaller value, we can then compute the h that would give the right value and then deduce the metric to be enforced :

$$\mathcal{M} = \frac{I_d}{h^2}\,,$$

with

$$h^2 = \frac{8\,\varepsilon}{M}\,.$$

In practice, when segment $I = [a,b]$ is analyzed, it is also of interest to look at the neighboring segments so as to guess M. Indeed, in practice, the problem is to find this maximum. The use of a parabola to simulate the function u can then lead to a solution from which the expected size h can be deduced.

Extension to a two-dimensional solution. We also consider a P^1 interpolation in two dimensions. The desired function u is approached by a solution $\Pi_h u$ computed at the triangles vertices of the mesh. Then, we look at the gap (or any other norm) between u et $\Pi_h u$ in each triangle. There are *a priori* two ways of controlling this error :

- either via a control by the edges and we come back to the previous discussion with, however, a second derivative (the Hessian) reduced to these edges,
- or via a control by the gap (or more precisely the L_∞ norm) between the triangle corresponding to the three known values and the (presumed) surface going through these three points. We then face a similar problem to that of the surface meshing.

The first solution is obviously quite rough. By analogy with a surface meshing problem, it consists of controlling the gap between a mesh triangle and the surface based on the sole evaluation of the relative gaps between the triangle edges and the surface. However, this solution gives an initial idea of the control.

The other approach is clearly smarter. It corresponds to what has been suggested in one dimension and, in its principle, can be seen as the direct extension of this approach.

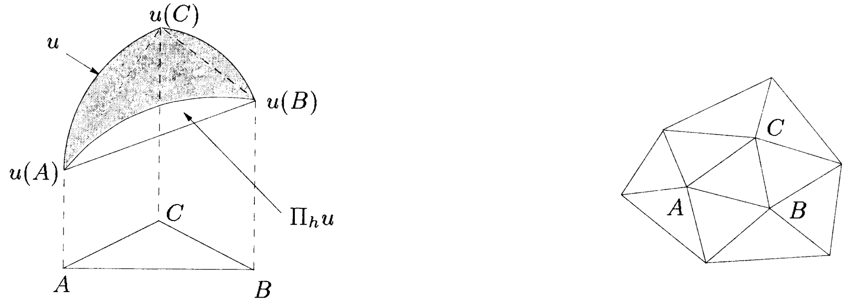

Figure 10.7: *Piecewise linear interpolation in two dimensions. Left, are shown the triangle $K = [A, B, C]$, $u(A), u(B)$ and $u(C)$, the function $\Pi_h u$, the triangle $[u(A), u(B), u(C)]$ and the function presumed u. Right, the triangle ABC and its neighbors are shown.*

Let us consider a function f from K to $\mathbb{R}$, where K is an interval of $\mathbb{R}^2$, actually the triangle of vertices A, B and C. For such a function, denoted $f(X)$, we look at how Relation (10.22) writes. We vary X over the triangle K. Under the same assumptions as previously, the local analysis based on a development around X, as seen from A, leads to :

$$f(A) = f(X) + \langle \overrightarrow{XA}\,, \nabla f(X)\rangle + \frac{1}{2}\langle \overrightarrow{XA}\,, H_f(X + t_1\overrightarrow{XA})\,\overrightarrow{XA}\rangle\,, \qquad (10.24)$$

with $\overrightarrow{XA}$ the displacement around A in the triangle K. In this expression, ∇ represents the gradient and $\mathcal{H}_f$ is the Hessian of f.

As in one dimension, we now consider as f the function $u - \Pi_h u$ and we are trying to write the previous relation with respect to u and $\Pi_h u$. Thus, we have :

$$(u-\Pi_h u)(A) = (u-\Pi_h u)(X) + \langle \overrightarrow{XA}\,, \nabla(u-\Pi_h u)(X)\rangle + \frac{1}{2}\langle \overrightarrow{XA}\,, H_u(X+t_1\overrightarrow{XA})\,\overrightarrow{XA}\rangle\,, \qquad (10.25)$$

as $\mathcal{H}_f = \mathcal{H}_u$. Now, let us turn to the term in ∇. To this end, let us assume that X where the extremum occurs falls[6] inside K. Then,

$$\nabla(u - \Pi_h u)(X) = 0\,,$$

[6] If not, X necessarily belongs to an edge of K, and the one-dimensional result holds.

and, the above relation reduces to :

$$0 = (u - \Pi_h u)(X) + \frac{1}{2}\langle \overrightarrow{XA}\,, H_u(X + t_1\overrightarrow{XA})\,\overrightarrow{XA}\rangle\,. \qquad (10.26)$$

Now, we write similar expressions for the same X as now expressed from B and C.

$$0 = (u - \Pi_h u)(X) + \frac{1}{2}\langle \overrightarrow{XB}\,, H_u(X + t_2\overrightarrow{XB})\,\overrightarrow{XB}\rangle\,,$$

$$0 = (u - \Pi_h u)(X) + \frac{1}{2}\langle \overrightarrow{XC}\,, H_u(X + t_3\overrightarrow{XC})\,\overrightarrow{XC}\rangle\,.$$

Adding these three relations leads to :

$$0 = 3(u-\Pi_h u)(X) + \frac{1}{2}\langle \overrightarrow{XA}\,, H_u(X+t_1\overrightarrow{XA})\,\overrightarrow{XA}\rangle + \frac{1}{2}\langle \overrightarrow{XB}\,, H_u(X+t_2\overrightarrow{XB})\,\overrightarrow{XB}\rangle$$

$$+\frac{1}{2}\langle \overrightarrow{XC}\,, H_u(X + t_3\overrightarrow{XC})\,\overrightarrow{XC}\rangle\,.$$

Let now M be such that :

$$M = max_{Y\in K}\left(\max_{\overrightarrow{vec}\in R^2} \frac{|\langle \overrightarrow{vec}\,, H_u(Y)\overrightarrow{vec}\rangle|}{||\overrightarrow{vec}||^2}\right)\,.$$

Then :

$$|(u - \Pi_h u)(X)| \le \frac{1}{6}\left(||\overrightarrow{AX}||^2 + ||\overrightarrow{BX}||^2 + ||\overrightarrow{CX}||^2\right) M\,.$$

Each point X of K can be written using a linear combination of A, B and C :

$$X = \lambda_a\, A + \lambda_b\, B + \lambda_c\, C\,,$$

with $\lambda_a + \lambda_b + \lambda_c = 1$. It is easy to see that :

$$\overrightarrow{AX} = \lambda_b\overrightarrow{AB} + \lambda_c\overrightarrow{AC}\,,$$

$$\overrightarrow{BX} = \lambda_c\overrightarrow{BC} + \lambda_a\overrightarrow{BA}\,,$$

$$\overrightarrow{AX} = \lambda_a\overrightarrow{CA} + \lambda_b\overrightarrow{CB}\,,$$

and thus :

$$||\overrightarrow{AX}||^2+||\overrightarrow{BX}||^2+||\overrightarrow{CX}||^2 \le (\lambda_a^2+\lambda_b^2)||\overrightarrow{AB}||^2 + (\lambda_a^2+\lambda_c^2)||\overrightarrow{AC}||^2 + (\lambda_b^2+\lambda_c^2)||\overrightarrow{BC}||^2$$

$$+2\,\lambda_a\lambda_b\,|\langle \overrightarrow{CA}\,, \overrightarrow{CB}\rangle| + 2\,\lambda_a\lambda_c\,|\langle \overrightarrow{BA}\,, \overrightarrow{BC}\rangle| + 2\,\lambda_b\lambda_c\,|\langle \overrightarrow{AB}\,, \overrightarrow{AC}\rangle|\,.$$

Let L be the length of the largest edge in K, then :

$$||\overrightarrow{AX}||^2+||\overrightarrow{BX}||^2+||\overrightarrow{CX}||^2 \le 2\,\left((\lambda_a^2 + \lambda_b^2 + \lambda_c^2) + \lambda_a\lambda_b + \lambda_a\lambda_c + \lambda_b\lambda_c\right)\, L^2\,.$$

It is easy to see that the extremum is obtained for :

$$\lambda_a = \lambda_b = \lambda_c = \frac{1}{3}\,,$$

and therefore, we have :

$$|(u - \Pi_h u)(X)| \le \frac{2}{9} L^2 M . \qquad (10.27)$$

The majorant that we give below is certainly not optimal and, in specific, it is isotropic. To improve this majoration, we go back to Formula (10.26).

To this end, we assume that X, the point where the maximum holds, is closer to A than to the two other vertices of K. Then, we write :

$$0 = (u - \Pi_h u)(X) + \frac{1}{2}\langle \overrightarrow{AX} , H_u(X')\, \overrightarrow{AX} \rangle , \qquad (10.28)$$

where X' lies on AX. Let us introduce A' the point intersection of AX with BC the side opposite vertex A. Then, due to the assumption about X, we have $\lambda \le \frac{2}{3}$ where λ is such that :

$$\overrightarrow{AX} = \lambda\, \overrightarrow{AA'} .$$

The above relation becomes :

$$|(u - \Pi_h u)(X)| = \frac{1}{2}\, ||\langle \overrightarrow{AX} , H_u(X')\, \overrightarrow{AX} \rangle || ,$$

$$|(u - \Pi_h u)(X)| = \frac{\lambda^2}{2}\, ||\langle \overrightarrow{AA'} , H_u(X')\, \overrightarrow{AA'} \rangle || ,$$

thus,

$$|(u - \Pi_h u)(X)| \le \frac{2}{9}\, ||\langle \overrightarrow{AA'} , H_u(X')\, \overrightarrow{AA'} \rangle || \qquad (10.29)$$

holds. As a consequence, we have obtained an anisotropic estimate. Note that the same reasoning can be made for B and C and a combination of the corresponding results is used to find the metric information needed for error control and mesh adaptation.

From a geometric point of view, if X is a point of K, the point $(X, \Pi_h u(X))$ covers the triangle $[u(A), u(B), u(C)]$ while the point $(X, u(X))$ covers the (unknown) "surface" u. As u is supposed sufficiently regular over K, we will replace this surface by a paraboloid. Hence, a method providing the desired result consists in :

- constructing a paraboloid going through the points $\Pi_h u(A) = u(A)$, $\Pi_h u(B) = u(B)$ and $\Pi_h u(C) = u(C)$ and
- this surface being known, evaluating $\mathcal{H}$ on K and finding the desired majorants.

Therefore we vary X to see the majoration values needed to define a suitable $\mathcal{H}^*$ in K using the various $\mathcal{H}$ we have computed[7]. We deduce the values of λ_1

[7] Note that this step is not obvious.

and of λ_2 that correspond to the principal directions of this matrix. The required metric consists in making sure that, given ε, each point X is such that :

$$\langle \overrightarrow{AX}, |\mathcal{H}^*|\, \overrightarrow{AX} \rangle = \varepsilon .$$

We then find a metric of the form :

$$\mathcal{M} = \mathcal{R} \begin{pmatrix} |\lambda_1| & 0 \\ 0 & |\lambda_2| \end{pmatrix} \mathcal{R}^{-1} ,$$

which, in the coordinate system defined by $\mathcal{R}$, corresponds to the ellipse :

$$\frac{|\lambda_1|}{\varepsilon} x^2 + \frac{|\lambda_2|}{\varepsilon} y^2 = 1 .$$

If X covers or is internal to this ellipse, the interpolation error is smaller than or equal to ε in any direction.

Extension to a three-dimensional solution. The same reasoning applies and the conclusions, in terms of the control, remain unchanged. The constant of error is then $\frac{9}{32}$. However, the calculations are considerably more technical and the geometric interpretation relies on a surface of $\mathbb{R}^4$ associated with the four vertices of a tetrahedron.

Remark 10.6 *Notice, to conclude this discussion, that the control of the interpolation error suggested here is purely geometric. It does not involve the physic of the problem via its operator. Other operators analyze this point of view. Moreover, other norms may be chosen for the error.*

Hessian computation. Finally, we briefly mention one of the potential numerical problems. One of the difficulties is actually to compute the Hessians involved in the estimate. Such a calculation can be carried out by inventing a surface assumed to represent the desired solution or by more directly numerical methods (Green's formula, for example).

More precisely, this approach using *generalized finite differences* is based on Green's formula :

$$\int_\Omega u\, \partial_i v \, d\Omega = - \int_\Omega \partial_i u\, v d\Omega + \int_\Gamma u\, v\, \nu_i \, d\Gamma ,$$

with the usual notations ([Ciarlet-1991]). Setting $u = \partial_i u$, we obtain :

$$\int_\Omega \partial_i u\, \partial_i v \, d\Omega = - \int_\Omega \partial^2_{ii} u\, v d\Omega + \int_\Gamma \partial_i u\, v\, \nu_i \, d\Gamma ,$$

that will allow us to find the diagonal coefficients of the Hessian of u and, similarly, setting $u = \partial_j u$, we find :

$$\int_\Omega \partial_j u\, \partial_i v \, d\Omega = - \int_\Omega \partial^2_{ij} u\, v d\Omega + \int_\Gamma \partial_j u\, v\, \nu_i \, d\Gamma ,$$

which leads to the non-diagonal coefficients of the Hessian.

For an internal point, the boundary term is null and the integral is calculated on the set of elements sharing the vertex where the Hessian is sought. Given some assumptions (Hessian constant per element), we find, for example for the coefficient $_{ii}$:

$$\partial_{ii} u = -\frac{\sum_k \alpha_k \, \alpha'_k \, S_k}{\sum_k V_k},$$

where :

- α_k is the derivative in i of the interpolation of the function u (a plane here, in P^1, the plane going through the points $u(A), u(B)$ and $u(C)$ where A, B and C are the vertices of the considered triangle),
- α'_k is the derivative in i of the surface defined over the triangle considered of null height except at the vertex concerned where the height is equal to 1,
- S_k is the triangle area of index k,
- V_k is the volume of the tetrahedron having the triangle of index k for base and corresponding to the hat function v_k for which the value is 1 at the considered vertex and 0 at the other vertices.

Similarly, we evaluate the terms in $_{ij}$. If the point at which the Hessian is sought is a boundary point, we compute the boundary contribution in the same fashion or, simply, we use an average of the Hessians at the neighboring internal vertices of the current point.

Problem with multiple unknowns. Metric intersection. When the physical problem has many unknowns that are all used to control the adaptation, each of them is used to construct a metric. We therefore face a context in which several fields of metrics are given. The previous section enables us to retrieve the case of a sole metric map, using the metric intersection scheme.

On the other hand, for surface meshes, the geometric metric (of the radii of curvature, for instance) can be intersected with one (or several) computational metrics. Thus, the geometric approximation of the surface can be preserved.

Mesh adaptation. Mesh adaptation (using a h-method, Chapter 21) can be based on the same principles as discussed previously, *i.e.*, by controlling the interpolation error. Once a solution has been computed, it is analyzed using an error estimate. The previous method gives a way of carrying out this analysis by noticing that the local analysis performed here is meaningless unless a more global analysis justifies its full validity.

Chapter 11

Differential geometry

Introduction

Mesh generation of curves and surfaces[1] is an operation known to be tedious to carry out, in a robust fashion, in the context of numerical simulations based on a finite element method as well as in other types of applications. The accuracy of the results in finite element numerical simulations is partly related to the quality of the geometric approximation (*i.e.*, the mesh). Therefore, the mesh of the boundary of a two- or three-dimensional arbitrary domain must have certain properties that are directly related to the geometry it represents.

The construction of a mesh of a curve (resp. surface) requires, in particular, knowledge of the local intrinsic characteristics of the curve (resp surface) such as the curvature, the normal(s), the tangent (the tangent plane), etc. at any point of this geometric support. These geometric characteristics have a significance that will be made precise in this chapter. This analysis is based in practice on a limited expansion of the function, γ or σ, which gives a local approximation of the corresponding curve or surface. The local behavior and the main features of the function can be deduced from this approximation. Thus, in a mesh generation context, the analysis of curves and surfaces can be deduced from γ or σ as well as from their successive derivatives.

Differential geometry was introduced in the early 18th century and then established in the 19th century as a way of defining a general theoretical framework for the local study of curves and surfaces. The fundamental contribution of Gauss[2] consisted in using a parametric representation of the surfaces and in showing the intrinsic nature of the total curvature. The breakthrough came with Riemann[3] who gave a global mathematical definition of curves and surfaces, introducing notably the notion of n-dimensional manifold.

[1] or, more generally, any mesh of the boundary of a domain.

[2] *Disquisitiones circa superficies curvas* (1827).

[3] *Sur les hypothèses qui servent de fondement à la géométrie* (1854).

★
★ ★

The purpose of this chapter is to review the elementary notions of differential geometry necessary to understand the chapters related to the modeling as well as the mesh generation of curves and surfaces (Chapters 12 to 15). If some of these notions may appear obvious to the initiated reader[4], we believe it advisable to recall these results so as to introduce the terminology and the notations that will be used subsequently. This chapter should not be considered as a substitute for the various courses and references in this domain, in which the reader may find the proofs of the main results given here ([doCarmo-1976], [Lelong-Ferrand,Arnaudies-1977], [Berger-1978], [Farin-1997], among others).

We limit ourselves here to the study of curves and surfaces embedded in the Euclidean space in two or three dimensions. The definitions require the use of the implicit function theorem. Therefore, we introduce the notion of a parameterized arc, and we conduct a local study. We also define surfaces and we introduce the two fundamental forms and the total curvature which plays an important role in the local study (it indicates the position of the surface with respect to its tangent plane) as well as the global curvature (Euler-Poincaré characteristic) of the surfaces. Finally, the last section is devoted to the practical aspects of the calculations related to curves and surfaces, in particular, the approximate resolution of non-linear problems.

11.1 Metric properties of curves and arcs

This section briefly recalls the different features used for the study of curves. In particular, we outline the notions of curvilinear abscissis, of arc length and we introduce the tangent and normal vectors as well as the Frénet frame. These definitions will allow us to compute quantities like the local curvature, the radius of curvature, the osculating circle as well as the local torsion and the relevant radius of torsion. Table 11.1, given at the end of the section devoted to curves, contains the values of the main geometric features encountered.

In the remainder of the chapter, we denote by $\mathcal{E}^d$ a Euclidean affine space of dimension d and by E^d the corresponding vector space (in practice, we consider that $E = \mathbb{R}$). We will study the properties of curves and geometric arcs of $\mathcal{E}^d$ related to the data of this Euclidean structure.

Let us recall that the structure $\mathcal{E}^3$ of Euclidean space of $\mathbb{R}^3$ is defined by the choice of the usual dot product (Chapter 10), for which the canonical basis :

$$[e_1 = (1,0,0)\,, e_2 = (0,1,0)\,, e_3 = (0,0,1)]$$

is orthonormal. The norm of a vector X of components (x, y, z) in the space $\mathbb{R}^3$ is simply :

$$||X|| = \sqrt{x^2 + y^2 + z^2}\,.$$

[4] Who might prefer to skip this chapter and go directly on to the next one.

11.1.1 Arc length

We recall that a curve Γ of a normed vector space E corresponds to a continuous application $\gamma : I \longrightarrow E$, defined on an interval I, which associates to the parameter $t \in I$ the value $\gamma(t)$. A curve is of class C^k if the application γ is of class C^k. If in addition, the interval I is compact, the curve Γ is said to be compact.

We will define the notion of length in an arbitrary normed vector space E. To this end, we introduce an approximation of the curve (the arc) γ by a set of *inscribed polygonal lines*.

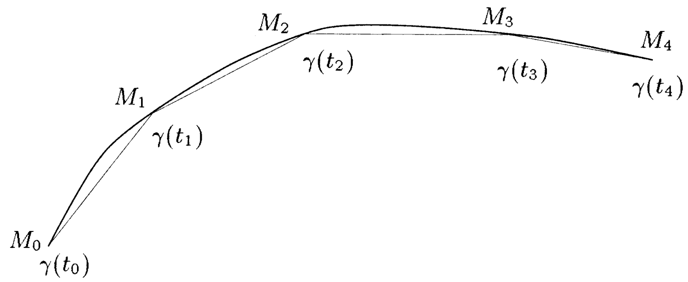

Figure 11.1: *Approximation of a curve Γ described by a function γ using an inscribed polygonal line.*

General notion of a rectifiable arc. With any subdivision $sub = (t_0, t_1, ..., t_p)$ of $[a, b]$ (with $a = t_0$ and $b = t_p$, $t_{i+1} \geq t_i$) into p segments, we associate the *polygonal line* $(M_0, M_1, ..., M_p)$ having $M_i = \gamma(t_i)$ as vertices. This subdivision is said to be *inscribed* into Γ.

Definition 11.1 *The length of this line is the number :*

$$L_{sub,\Gamma} = \sum_{i=0}^{p-1} ||\overrightarrow{M_i M_{i+1}}|| = \sum_{i=0}^{p-1} ||\gamma(t_{i+1}) - \gamma(t_i)||\,.$$

An arc Γ of a normed vector space is said to be *rectifiable* if the upper bound $L(\Gamma)$ of the lengths $L_{sub,\Gamma}$ of the polygonal lines inscribed in Γ is *finite*. In this case, $L(\Gamma)$ is a positive real number called the *length* of Γ. Obviously we have :

$$L(\Gamma) \geq ||\gamma(b) - \gamma(a)||\,.$$

Remark 11.1 *The length of a rectifiable arc depends on the norm chosen.*

Remark 11.2 *The fact that a curve is continuous does not necessarily imply that its length is bounded. A counter-example is that of fractal curves.*

Theorem 11.1 *In a complete normed vector space E, any compact arc Γ (of endpoints $A = \gamma(a)$ and $B = \gamma(b)$) of class C^k $(k \geq 1)$ is* rectifiable. *If in* *addition, $\gamma : [a,b] \longrightarrow E$ is a parameterization of Γ, the length of Γ is :*

$$L(\Gamma) = \int_a^b ||\gamma'(t)||dt\,. \tag{11.1}$$

Proof. The proof, left as an exercise, consists in showing that Γ is rectifiable, then posing :

$$\Delta_{sub} = \int_a^b ||\gamma'(t)||dt - L_{sub,\Gamma} = \sum_{i=0}^{p-1}\left(\int_{t_i}^{t_{i+1}} ||\gamma'(t)||dt - ||\gamma(t_{i+1}) - \gamma(t_i)||\right),$$

it is sufficient to prove that, for each $\varepsilon > 0$, there exists a subdivision *sub* such that $\Delta_{sub} \leq \varepsilon$. □

In the affine Euclidean space $\mathcal{E}^d$, with the usual norm, we then find, using Relation (11.1) :

$$L(\Gamma) = \int_a^b \sqrt{\gamma_1'^2(t) + ... + \gamma_d'^2(t)}dt \text{ that is, for } d = 3\,, \int_\gamma \sqrt{dx^2 + dy^2 + dz^2}\,.$$

For example in $\mathbb{R}^2$, the length L of the circle of center O and of radius ρ represented by $x = \rho\cos t\,, y = \rho\sin t\,, t \in [0, 2\pi]$, is equal to (obviously) :

$$L = \int_0^{2\pi} \rho\sqrt{\sin^2 t + \cos^2 t}dt = \rho\int_0^{2\pi} dt = 2\pi\rho\,.$$

11.1.2 Curvilinear abscissa, normal parameters and characteristics

Definition 11.2 *Let Γ be an arc of class C^1. We call* normal parameterization *of Γ any parameterization $\gamma : I \longrightarrow \mathcal{E}^d$ such that, $\forall t \in I$, we have : $||\gamma'(t)|| = 1$.*

If the arc Γ is simple and *oriented* in the sense of the ascending t, the number :

$$s(t) = \int_{t_0}^t ||\gamma'(\theta)||d\theta$$

is called *curvilinear abscissa* of the point $M = \gamma(t)$, measured from the point M_0, that is $\gamma(t_0)$, the *origin*. In other words, the length of the portion of curve joining two points of this curve can be expressed as a sum of arc lengths. Hence, we are able to write :

$$||\gamma'(t)|| = s'(t)\,, \qquad \text{thus} \quad ds = ||\gamma'(t)||\,dt\,.$$

If the parameterization is normal, we have $||\gamma'(t)|| = 1$ and thus, $ds = dt$. In other words, for such parameterizations, s and t are identical.

Tangent vector. Let Γ be a regular oriented arc of class D^2 defined by a normal parameterization $\bar{\gamma} : I \longrightarrow \mathcal{E}^d$, $s \longmapsto \bar{\gamma}(s)$.

Definition 11.3 *The function $\vec{\tau} : I \longrightarrow \mathcal{E}_d$, $s \longmapsto \bar{\gamma}'(s)$ defines the* unit tangent vector *to Γ at any s.*

This tangent vector indicates the direction of the tangent to the curve.

Remark 11.3 *If $M = \bar{\gamma}(s)$ is not a simple point of Γ, the vector $\vec{\tau}(s)$ admits a value to the left and a value to the right that are different.*

If we consider a regular parameterization, such that, $||\gamma'(t)|| \neq 0$, for each t, the unit tangent vector $\vec{\tau}(t)$ to Γ at t is defined by :

$$\vec{\tau}(t) = \frac{d\gamma(t)}{ds} = \frac{d\gamma(t)}{dt}\frac{dt}{ds} = \frac{\gamma'(t)}{||\gamma'(t)||}$$

and, if the parameterization is normal, we simply have $\vec{\tau} = \gamma'(s)$.

Principal normal. Curvature. The hyperplane denoted as Π_n going through M and orthogonal to $\vec{\tau}(s)$ is called *normal* to Γ in M. This plane (Figure 11.4) contains the various normals to the curve. By extension, any line going through M and orthogonal to $\vec{\tau}(s)$ is a *normal* to Γ in M.

As in the case of a normal parameterization, $||\bar{\gamma}'(s)|| = 1$, a derivation shows that : $\langle\bar{\gamma}'(s), \bar{\gamma}''(s)\rangle = 0$ and thus, these two vectors are orthogonal to each other. Differently written, we have the relationship :

$$\langle\vec{\tau}(s), \frac{d\vec{\tau}(s)}{ds}\rangle = 0\,.$$

Therefore, there exists a (scalar) function, $C : s \longmapsto C(s)$, such that :

$$\frac{d\vec{\tau}(s)}{ds} = C\,\vec{\nu}(s)\,,$$

where $\vec{\nu}(s)$ is the vector supported by $\frac{d\vec{\tau}(s)}{ds}$, which is a unit one. This vector is called unit oriented *normal vector* to Γ. The function $C(s)$ is the *curvature* of Γ in s.

The curvature function of Γ in s can be also expressed by the relation :

$$C(s) = ||\bar{\gamma}''(s)||\,.$$

Remark 11.4 *In the case of planar curves, as with the oriented plane, we can define an* algebraic curvature. *To this end, we define, from $\vec{\tau}(s)$, the vector $\vec{\nu}_1(s)$ by rotation of value $\pi/2$ and we have $\frac{d\vec{\tau}(s)}{ds} = C\vec{\nu}_1(s)$, which implies that C is signed.*

We note that the curvature is null[5] if and only if the point $M = \gamma(s)$ is an *inflection* point. If the arc Γ has no inflection point, the vector $\vec{\nu}(s)$ is defined at any s.

Moreover, the *radius of curvature* of Γ in M is the number $\rho(s) = \frac{1}{C(s)}$. The point O defined by $\overrightarrow{MO} = \rho(s)\,\vec{\nu}(s)$ is the *center of curvature* of Γ in M.

If Γ is a simple regular arc having the point O as the center of curvature in M, the circle of center O passing through M and contained in plane Π defined by the basis $[\vec{\tau}, \vec{\nu}]$ is the *osculating circle* of the curve in M (Figure 11.2, right-hand side). The latter plane is the *osculating plane* to Γ in M.

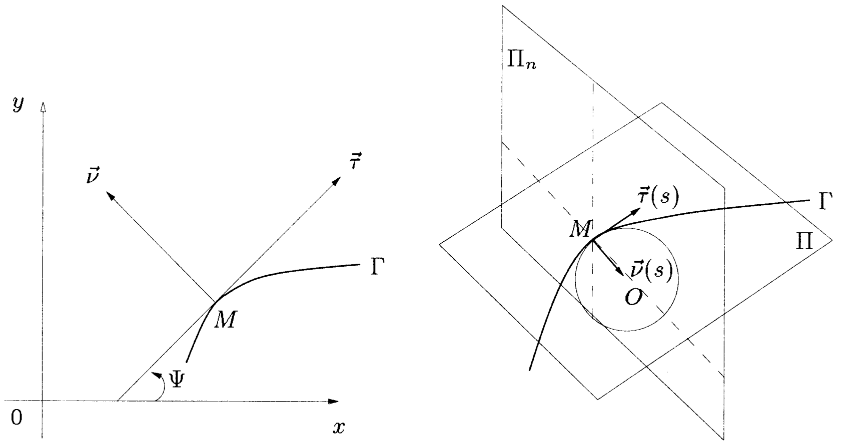

Figure 11.2: *Left-hand side, unit normal vector to the arc Γ in M. Right-hand side, the intersection of planes Π and Π_n is a line supporting the normal $\vec{\nu}$ and containing the point O, the center of curvature to Γ in M.*

Arbitrary parameterization. Let γ be a parameterization of at least class D^2 defining an oriented arc Γ, which is regular and without any inflection point. We go back to a normal parameterization $\bar{\gamma}$ using the change of parameter $t \longmapsto s(t)$, such that $ds/dt = ||\gamma'(t)||$. As $\gamma(t) = \bar{\gamma}(s(t))$, we have :

$$\gamma'(t) = \bar{\gamma}'(s)\frac{ds}{dt}\,; \quad \gamma''(t) = \bar{\gamma}'(s)\frac{d^2(s)}{dt^2} + \bar{\gamma}''(s)\left(\frac{ds}{dt}\right)^2,$$

from which we have the formula (depending on $\vec{\tau}$ and on $\vec{\nu}$) :

$$\gamma''(t) = \frac{d^2 s}{dt^2}\vec{\tau}(s) + C(s)\left(\frac{ds}{dt}\right)^2 \vec{\nu}(s)\,, \tag{11.2}$$

that can also be expressed as follows :

$$\gamma''(t) = \frac{\langle \gamma'(t), \gamma''(t)\rangle}{||\gamma'(t)||}\vec{\tau}(s) \,+\, C(s)||\gamma'(t)||^2\vec{\nu}(s)\,.$$

[5]Except for the trivial case of lines in the plane !

Of all the normals, some have privileged directions. The intersection between Π and Π_n is a line oriented along the vector $\vec{\nu}(t)$ (Figure 11.2, right-hand side). This normal is of particular interest in the study of the curves, it contains in fact, as already mentioned, the center of curvature of the curve (see for instance, [Lelong-Ferrand,Arnaudies-1977]).

Practical calculation of the curvature. In practice, it is not always possible to express the curvature of the formula $C(s) = ||\bar{\gamma}''(s)||$ (*i.e.*, the expression of the normal parameterization can be too complex to be practical). The calculation will thus be based on the relation :

$$C = ||\gamma''(s)|| = \left\|\frac{d\vec{\tau}(s)}{ds}\right\|.$$

We then consider an arbitrary parameterization of Γ. If M denotes the point $\bar{\gamma}(s) = \gamma(t)$, with $s = s(t)$. We start from the relation :

$$\frac{d\vec{\tau}(s)}{ds} = \frac{d\vec{\tau}(t)}{dt}\frac{dt}{ds} = \frac{1}{||\gamma'(t)||}\frac{d\vec{\tau}(t)}{dt}\,,$$

but,

$$\frac{d\vec{\tau}(t)}{dt} = \frac{\gamma''(t)||\gamma'(t)|| - \frac{d}{dt}(||\gamma'(t)||)\,\gamma'(t)}{||\gamma'(t)||^2}\,,$$

as, $||\gamma'(t)|| = \sqrt{\langle\gamma'(t), \gamma'(t)\rangle}$, we have :

$$\frac{d}{dt}(||\gamma'(t)||) = \frac{\langle\gamma'(t), \gamma''(t)\rangle}{||\gamma'(t)||}\,,$$

hence :

$$\frac{d\vec{\tau}(t)}{ds} = \frac{1}{||\gamma'(t)||^3}\left(\gamma''(t)||\gamma'(t)|| - \gamma'(t)\frac{\langle\gamma'(t), \gamma''(t)\rangle}{||\gamma'(t)||}\right),$$

which finally leads to the well-known formula :

$$\frac{d\vec{\tau}(t)}{ds} = \frac{\gamma'(t) \wedge \gamma''(t)}{||\gamma'(t)||^3}\,, \tag{11.3}$$

where $\wedge$ represents the cross product (to obtain this result, we use the relation $(a \wedge b) \wedge c = \langle c, a\rangle\, b - \langle c, b\rangle\, a$). Thus we have,

$$C(t) = \frac{||\gamma'(t) \wedge \gamma''(t)||}{||\gamma'(t)||^3}\,. \tag{11.4}$$

Notice that in the case of a normal parameterization, this general expression gives the known result, that is :

$$C(s) = ||\gamma''(s)||\,. \tag{11.5}$$

Particular case. Let us assume that we are using Cartesian coordinates to calculate this. Let $(x(t), y(t))$ be the coordinates of $\gamma(t)$ in an orthonormal frame. We can write :

$$\frac{ds}{dt} = \sqrt{x'^2(t) + y'^2(t)}$$

and

$$\gamma'(t) \wedge \gamma''(t) = x'(t)y''(t) - y'(t)x''(t)$$

thus we deduce the algebraic curvature of Γ at the point $M = \gamma(t)$:

$$C = \frac{x'(t)y''(t) - y'(t)x''(t)}{\sqrt{x'^2(t) + y'^2(t)}^3}.$$

11.1.3 Frénet's frame and formula

The two orthogonal vectors $\vec{\tau}(s)$ and $\vec{\nu}(s)$ define the *osculating plane.* From these vectors, we define the *unit binormal vector* $\vec{b}(s)$ as :

$$\vec{b}(s) = \vec{\tau}(s) \wedge \vec{\nu}(s),$$

and the triple $[\vec{\tau}(s), \vec{\nu}(s), \vec{b}(s)]$ forms a basis that defines *Frénet's frame* at the point M of abscissa s :

$$\mathcal{F}_{frenet} = [M, \vec{\tau}, \vec{\nu}, \vec{b}].$$

Notice that if the orientation of Γ changes, $\vec{\tau}$ and $\vec{\nu}$ are then both changed to their opposite.

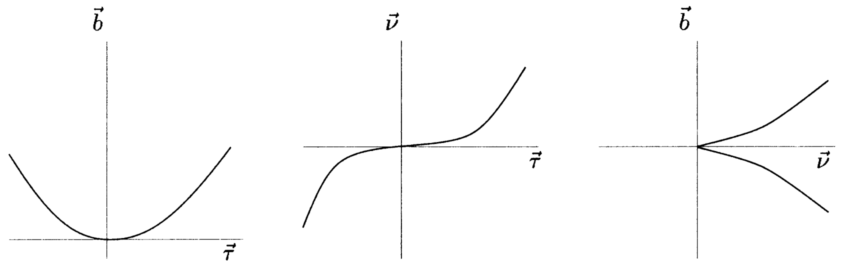

Figure 11.3: *Projections of a curve on the three planes of Frénet's frame. Left-hand side : regular point, middle : inflection point, right-hand side : cusp point.*

By derivating the relation $\langle \vec{\tau}(s), \vec{\nu}(s) \rangle = 0$, we obtain :

$$\left\langle \frac{d\vec{\tau}}{dt}(s), \vec{\nu}(s) \right\rangle + \left\langle \vec{\tau}(s), \frac{d\vec{\nu}}{ds}(s) \right\rangle = 0,$$

so, when introducing the curvature, the relation :

$$\left\langle \vec{\tau}(s), \frac{d\vec{\nu}(s)}{ds} \right\rangle = -C.$$

The two formulas :

$$\frac{d\vec{\tau}}{ds} = C\vec{\nu} \quad \text{et} \quad \frac{d\vec{\nu}}{ds} = -C\vec{\tau}, \tag{11.6}$$

are the *Frénet formulas* for the arc Γ.

11.1.4 Local behavior of a planar curve

Let Γ be a simple regular arc of a sufficient class, defined by an arbitrary parameterization γ or a normal parameterization $\bar{\gamma}$. We wish to study the behavior of the curve Γ in the vicinity of a point $M_0 = \gamma(t_0) = \bar{\gamma}(s_0)$. We go back to the case $t = s = 0$ using a translation. According to the form of the parameterization, we will study the local behavior of the curve in the local basis of the vectors $[\gamma', \gamma'']$ in $t = 0$ or in the Frénet frame $[\vec{\tau}, \vec{\nu}]$ in $s = 0$. So, we consider a small increase (a neighborhood) Δt or Δs in t or in s sufficiently small.

Applying a Taylor series at the first order to γ, we have :

$$\gamma(\Delta t) = \gamma(0) + \Delta t \gamma'(0) + \mathcal{O}(\Delta t^2)$$

and for $\bar{\gamma}$, we have similarly :

$$\bar{\gamma}(\Delta s) = \bar{\gamma}(0) + \Delta s \bar{\gamma}'(0) + \mathcal{O}(\Delta s^2),$$

with here, $\gamma'(0) = \vec{\tau}(0)$. This local study corresponds to an approximation of the curve using a line supported by the tangent at the point M_0. To show the gap between this approximation and the curve, we use a Taylor series at order 2 :

$$\gamma(\Delta t) = \gamma(0) + \Delta t \gamma'(0) + \frac{\Delta t^2}{2}\gamma''(0) + \mathcal{O}(\Delta t^3)$$

and for $\bar{\gamma}$, we have similarly :

$$\bar{\gamma}(\Delta s) = \bar{\gamma}(0) + \Delta s \bar{\gamma}'(0) + \frac{\Delta s^2}{2}\bar{\gamma}''(0) + \mathcal{O}(\Delta s^3),$$

with here, $\gamma'(0) = \vec{\tau}(0)$ and $\gamma''(0) = \vec{\nu}(0)/\rho(0) = C(0)\,\vec{\nu}(0)$ where $\rho(0)$ is the radius of curvature ($C(0)$ the curvature) at point M_0. In these expressions, the second order term measures the desired gap. This local study corresponds to an approximation of the curve by a parabola which, in the former case, is defined in a non orthonormal frame, and in the latter case, is defined in the orthonormal Frénet frame. The gap between this parabola and the curve can be obtained by pursuing the expansion at a higher order. A Taylor series at order 3 applied to γ gives, on the one hand :

$$\gamma(\Delta t) = \gamma(0) + \Delta t \gamma'(0) + \frac{\Delta t^2}{2}\gamma''(0) + \frac{\Delta t^3}{6}\gamma'''(0) + \mathcal{O}(\Delta t^4)$$

and, on the other hand :

$$\bar{\gamma}(\Delta s) = \bar{\gamma}(0) + \Delta s \bar{\gamma}'(0) + \frac{\Delta s^2}{2}\bar{\gamma}''(0) + \frac{\Delta s^3}{6}\bar{\gamma}'''(0) + \mathcal{O}(\Delta s^4).$$

As the local study is more intuitive in Frénet's frame and thus in the case of a normal parameterization, we will examine this case more carefully. Let us recall that $\bar{\gamma}'(0) = \vec{\tau}(0)$ and that $\bar{\gamma}''(0) = \vec{\nu}(0)/\rho(0) = C(0)\,\vec{\nu}(0)$ and let us express $\bar{\gamma}'''(0)$ in Frénet's frame :

$$\bar{\gamma}'''(s) = \frac{d\dfrac{\vec{\nu}(s)}{\rho(s)}}{ds} = -\frac{\vec{\tau}(s) + \rho'(s)\,\vec{\nu}(s)}{\rho^2(s)}.$$

Then, the previous limited expansion can be also expressed as follows :

$$\vec{\gamma}(\Delta s) = \vec{\gamma}(0) + \Delta s \vec{\tau}(0) + \frac{\Delta s^2}{2\,\rho(0)}\vec{\nu}(0) - \frac{\Delta s^3}{6\,\rho^2(0)}\left(\vec{\tau}(0) + \rho'(0)\vec{\nu}(0)\right) + \mathcal{O}(\Delta s^4)\,.$$

The term $\frac{\Delta s^3}{6\,\rho^2(0)}(\vec{\tau}(0) + \rho'(0)\vec{\nu}(0))$ measures the gap to the parabola.

The coordinates $x(\Delta s), y(\Delta s)$ of the point M of abscissa Δs are such that :

$$\left\{\begin{array}{lcl} x(\Delta s) & = & \Delta s - \dfrac{\Delta s^3}{6\,\rho^2(0)} + \mathcal{O}(\Delta s^4) \\ y(\Delta s) & = & \dfrac{\Delta s^2}{2\,\rho(0)} - \dfrac{\Delta s^3\,\rho'(0)}{6\,\rho^2(0)} + \mathcal{O}(\Delta s^4) \end{array}\right. .$$

Then,

- if, in $s = 0$, the curvature is null $C(0) = 0$ (*i.e.*, if the radius of curvature $\rho(0)$ is infinite), we have the case of an inflection point (cf. Figure 11.3, middle). The arc crosses its tangent at point M_0. Notice that a line segment has no inflection point.

- if $C(0) \neq 0$, the arc Γ is on the same side of its tangent as that of its center of curvature in M_0. We can specify the position of Γ by considering a circle Γ_a of radius $|a|$, tangent to Γ in M_0 defined by the normal parameterization :

$$X(s) = a \sin\left(\frac{s}{a}\right)\,, \quad Y(s) = a\left(1 - \cos\left(\frac{s}{a}\right)\right)\,.$$

In particular, for

$$a = \frac{1}{C(0)} = \rho(0)\,,$$

the circle Γ_a is the osculating circle to Γ. Using a limited expansion of the sine and cosine, we find :

$$x(\Delta s) - X(\Delta s) = \mathcal{O}(\Delta s^4) \quad \text{and} \quad y(\Delta s) - Y(\Delta s) = -\frac{\Delta s^3\,\rho'(0)}{6\,\rho^2(0)} + \mathcal{O}(\Delta s^4)\,.$$

So, if $\rho'(0) \neq 0$, the sign of $y(\Delta s) - Y(\Delta s)$ changes with that of Δs and the arc crosses its osculating circle in M_0.

Geometric interpretation of the osculating circle. Let us consider the circle Γ_a defined by the former parameterization, where a is an arbitrary scalar value. This circle passes through the point $M = (x(s), y(s))$ if and only if $a = \lambda(s)$, where

$$\lambda(s) = \frac{x^2(s) + y^2(s)}{2y(s)} = \frac{x^2(s)}{2y(s)} + \frac{1}{2}y(s)\,.$$

So, we can deduce :

$$\lim_{s\to 0} \lambda(t) = \lim_{t \to 0} \frac{x^2(t)}{2y(t)} = \frac{1}{C(0)} = \rho(0)\,.$$

Then, for each $M \in \Gamma$, close to M_0, there exists a unique circle Γ_M tangent to Γ in M_0 passing through M. The osculating circle to Γ in M_0 is the limit of Γ_M when M tends toward M_0 along Γ.

11.1.5 Arcs in $\mathbb{R}^3$. Frénet's frame and Serret-Frénet's formulas

Let us assume that the arc Γ has no inflection point (*i.e.*, $C(s) \neq 0, \forall s \in I$). *Frénet's frame* $[M, \vec{\tau}(s), \vec{\nu}(s), \vec{b}(s)]$ is, as already seen, the direct orthonormal frame of $\mathbb{R}^3$ defined by the vectors $\vec{\tau}(s)$, $\vec{\nu}(s)$and the vector $\vec{b}(s) = \vec{\tau}(s) \wedge \vec{\nu}(s)$ that is the *binormal vector* to Γ. The tangent plane passing through M, of main vectors $\vec{\nu}(s)$ and $\vec{b}(s)$ is called the *normal plane* to Γ in M and the plane passing through M, of main vectors $\vec{\tau}(s)$ and $\vec{b}(s)$ is called the *rectifying plane* to Γ in M (Figure 11.4, left-hand side).

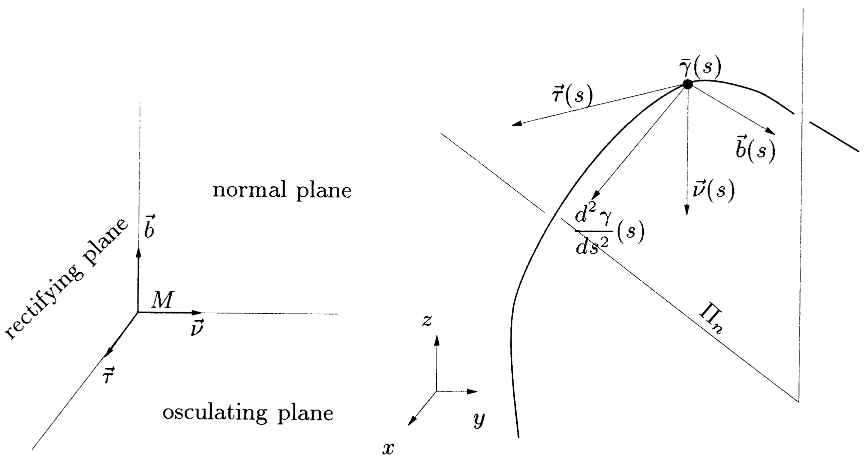

Figure 11.4: *Elements of differential geometry related to the parameter s at a point M of* Γ.

Serret-Frénet's formulas. Torsion. By assumption, the functions $\vec{\tau}$, $\vec{\nu}$ and $\vec{b}$ are such that :

$$||\vec{\tau}(s)|| = ||\vec{\nu}(s)|| = ||\vec{b}(s)|| = 1\,,$$

$$\langle \vec{\tau}(s), \vec{\nu}(s) \rangle = 0\,, \qquad \langle \vec{b}(s), \vec{\tau}(s) \rangle = 0\,, \qquad \langle \vec{b}(s), \vec{\nu}(s) \rangle = 0\,.$$

In order to establish Serret-Frénet's formulas, we will express the derivatives of these three vectors of Frénet's basis in this same basis. For $\vec{\tau}(s)$, we already know that :

$$\frac{d\vec{\tau}(s)}{ds} = C(s)\,\vec{\nu}(s)\,. \tag{11.7}$$

For $\vec{\nu}$, we write (and we look for a_0, a_1 and a_2) :

$$\frac{d\vec{\nu}(s)}{ds} = a_0\vec{\tau}(s) + a_1\vec{\nu}(s) + a_2\vec{b}(s)\,.$$

We obviously have $a_1 = 0$. To find a_0 we look at the dot product by $\vec{\tau}(s)$:

$$a_0 = \left\langle \frac{d\vec{\nu}(s)}{ds}, \vec{\tau}(s) \right\rangle = -\left\langle \frac{d\vec{\tau}(s)}{ds}, \vec{\nu}(s) \right\rangle = -C\,,$$

to find a_2, we consider the product by $\vec{b}(s)$:

$$a_2 = \left\langle \frac{d\vec{\nu}(s)}{ds}, \vec{b}(s) \right\rangle,$$

which is the opposite the *torsion*, $T(s)$. So, we have :

$$\frac{d\vec{\nu}(s)}{ds} = -C(s)\vec{\tau}(s) - T(s)\,\vec{b}(s)\,. \tag{11.8}$$

For $\vec{b}(s)$, we write again :

$$\frac{d\vec{b}(s)}{ds} = a_0\vec{\tau}(s) + a_1\vec{\nu}(s) + a_2\vec{b}(s)\,.$$

Here, $a_0 = 0$, $a_1 = T$ and $a_2 = 0$ and thus :

$$\frac{d\vec{b}(s)}{ds} = T(s)\vec{\nu}(s)\,. \tag{11.9}$$

By combining the Relations (11.7), (11.8) and (11.9), we obtain *Frénet*'s (or *Serret-Frénet*'s) formulas :

$$\frac{d\vec{\tau}}{ds} = C\vec{\nu}\,, \qquad \frac{d\vec{\nu}}{ds} = -C\vec{\tau} - T\vec{b}\,, \qquad \frac{d\vec{b}}{ds} = T\vec{\nu}\,. \tag{11.10}$$

the number $\rho(s) = 1/C(s)$ is called the *radius of curvature* of Γ at the point $M = \bar{\gamma}(s)$. If also, $T(s) \neq 0$, the number $R_T(s) = 1/T(s)$ is called the *radius of torsion* of Γ at the point M. With these two notations, Serret-Frénet's formulas can be written as :

$$\frac{d\vec{\tau}}{ds} = \frac{\vec{\nu}}{\rho}\,, \qquad \frac{d\vec{\nu}}{ds} = -\frac{\vec{\tau}}{\rho} - \frac{\vec{b}}{R_T}\,, \qquad \frac{d\vec{b}}{ds} = \frac{\vec{\nu}}{R_T}\,. \tag{11.11}$$

These relations indicate the fact that the matrix $\mathcal{M}$ of the vectors $d\vec{\tau}/ds$, $d\vec{\nu}/ds$, $d\vec{b}/ds$ in the basis $[\vec{\tau}(s), \vec{\nu}(s), \vec{b}(s)]$ is :

$$\mathcal{M} = \begin{bmatrix} 0 & -C(s) & 0 \\ C(s) & 0 & T(s) \\ 0 & -T(s) & 0 \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{-1}{\rho(s)} & 0 \\ \dfrac{1}{\rho(s)} & 0 & \dfrac{1}{R_T(s)} \\ 0 & \dfrac{-1}{R_T(s)} & 0 \end{bmatrix}. \tag{11.12}$$

This matricial relation is useful when looking at the behavior of the curve in space when the parameter changes.

11.1.6 Computing curvature and torsion

If Γ is defined by a normal parameterization, Frénet's formulas allow us to evaluate the curvature and torsion of Γ by means of the derivatives $\bar{\gamma}'$, $\bar{\gamma}''$ and $\bar{\gamma}'''$. We have already seen, in fact, the case of the curvature. For a normal parameterization, it is Relation (11.5) :

$$C(s) = ||\gamma''(s)||\,.$$

For an arbitrary parameterization, it is Relation (11.4) :

$$C(t) = \frac{||\gamma'(t) \wedge \gamma''(t)||}{||\gamma'(t)||^3}\,.$$

The torsion is then obtained using the previous formulas. The easiest way of computing the torsion is to start from Relation (11.8) :

$$\frac{d\vec{\nu}(s)}{ds} = -C(s)\vec{\tau}(s) - T(s)\,\vec{b}(s)$$

and to apply the dot product with $\vec{b}(s)$. Hence, we find (omitting the sign) :

$$T(s) = \left\langle \frac{d\vec{\nu}(s)}{ds}, \vec{b}(s) \right\rangle = \left\langle \frac{d\vec{\nu}(s)}{ds}, \vec{\tau}(s) \wedge \vec{\nu}(s) \right\rangle = det\left| \frac{d\vec{\nu}(s)}{ds}, \vec{\tau}(s), \vec{\nu}(s) \right|.$$

We have $\vec{\tau}(s) = \bar{\gamma}'(s)$, $\vec{\nu}(s) = \frac{1}{C(s)}\frac{d\vec{\tau}(s)}{ds} = \frac{1}{C(s)}\bar{\gamma}''(s)$ and it is then sufficient to express $\frac{d\vec{\nu}(s)}{ds}$ to find the desired result.

$$\frac{d\vec{\nu}(s)}{ds} = \frac{d}{ds}\left(\frac{1}{C(s)}\bar{\gamma}''(s)\right),$$

however

$$\frac{d}{ds}\left(\frac{1}{C(s)}\bar{\gamma}''(s)\right) = \frac{1}{C(s)^2}\left(C(s)\bar{\gamma}'''(s) - \bar{\gamma}''(s)\frac{dC(s)}{ds}\right)$$

and

$$\frac{dC(s)}{ds} = \frac{1}{C(s)}\langle\bar{\gamma}'''(s), \bar{\gamma}''(s)\rangle = 0\,,$$

then, $\frac{d\vec{\nu}(s)}{ds}$ can be reduced to $\frac{1}{C(s)}\gamma'''(s)$ and the torsion is (omitting the sign) :

$$T(s) = \frac{1}{C(s)^2} det\,|\bar{\gamma}'''(s)\;\;\bar{\gamma}'(s)\;\;\bar{\gamma}''(s)| = \frac{1}{C(s)^2} det\,|\bar{\gamma}'(s)\;\;\bar{\gamma}''(s)\;\;\bar{\gamma}'''(s)|\,. \tag{11.13}$$

We find again the relation (without the sign) :

$$det\,|\bar{\gamma}'(s), \bar{\gamma}''(s), \bar{\gamma}'''(s)| = -C^2(s)\,T(s)\,.$$

Remark 11.5 *These formulas are not always obvious to compute. If the function $ds/dt = ||\gamma'(t)||$ admits a simple expression, it may be interesting to compute the components of the vectors $\vec{\tau}, \vec{\nu}$ and $\vec{b}$ at the point M, which is equivalent to applying Frénet's formulas without specifying the normal parameter.*

So, we can express the torsion as a function of t. We have :

$$T(t) = det \left| \frac{d\vec{\nu}(t)}{ds} \ \vec{\tau}(t) \ \vec{\nu}(t) \right| .$$

In this determinant, we now have :

$$\vec{\tau}(t) = \frac{\gamma'(t)}{||\gamma'(t)||},$$

$$\vec{\nu}(t) = \frac{1}{C(s)} \frac{d\vec{\tau}(t)}{ds} = \frac{1}{C(s)} \frac{\gamma'(t) \wedge \gamma''(t)}{||\gamma'(t)||^3} = \frac{\gamma'(t) \wedge \gamma''(t)}{||\gamma'(t) \wedge \gamma''(t)||},$$

and for $\frac{d\vec{\nu}(t)}{ds}$, we obtain :

$$\frac{d\vec{\nu}(t)}{ds} = \frac{\vec{\nu}'(t)}{||\gamma'(t)||} .$$

It remains to express $\vec{\nu}'(t)$. We have :

$$\vec{\nu}'(t) = \left(\frac{1}{C(s)} \frac{\gamma'(t) \wedge \gamma''(t)}{||\gamma'(t)||^3} \right)',$$

and the calculation gives :

$$\vec{\nu}'(t) = \frac{\gamma'(t) \wedge \gamma'''(t)}{||\gamma'(t) \wedge \gamma''(t)||},$$

so, we find :

$$T(t) = det \left| \frac{1}{||\gamma'(t)||} \frac{\gamma'(t) \wedge \gamma'''(t)}{||\gamma'(t) \wedge \gamma''(t)||} \ \frac{\gamma'(t)}{||\gamma'(t)||} \ \frac{\gamma'(t) \wedge \gamma''(t)}{||\gamma'(t) \wedge \gamma''(t)||} \right|,$$

which gives the expression of the torsion :

$$T(t) = \frac{det|\gamma'(t), \gamma''(t), \gamma'''(t)|}{||\gamma'(t) \wedge \gamma''(t)||^2} . \tag{11.14}$$

Use of the tangent indicatrix. The indicatrix of the tangents γ_1 is defined by the parameterization :

$$\gamma_1 : I \longrightarrow E^3 , \quad t \longmapsto \frac{\gamma'(t)}{||\gamma'(t)||} .$$

The unit tangent vector to γ_1 is $\vec{\tau}_1 = \vec{\nu}$ and the curvature C of γ is :

$$C(s) = ||\gamma_1'(t)|| \frac{dt}{ds} .$$

Summary table. Table 11.1 summarizes the various vector functions and definitions introduced in this section, along with their notations.

function	notat.	depending on t	depending on s
curvilinear abscissa	s	$s(t) = \int_{t_0}^{t} \|\|\gamma'(u)\|\| \, du$	
tangent vector	$\vec{\tau}$	$\frac{\gamma'(t)}{\|\|\gamma'(t)\|\|}$	
curvature	$C = \frac{1}{\rho}$	$\frac{\|\|\gamma'(t) \wedge \gamma''(t)\|\|}{\|\|\gamma'(t)\|\|^3}$	$\|\|\bar{\gamma}''(s)\|\|$
normal vector	$\vec{\nu}$	$\frac{1}{C(t)} \frac{d\vec{\tau}(t)}{dt}$	$\frac{\bar{\gamma}''(s)}{\|\|\bar{\gamma}''(s)\|\|}$
torsion	T	$\frac{det\|\gamma'(t), \gamma''(t), \gamma'''(t)\|}{\|\|\gamma'(t) \wedge \gamma''(t)\|\|^2}$	$\frac{det\|\bar{\gamma}'(s), \bar{\gamma}''(s), \bar{\gamma}'''(s)\|}{C^2(s)}$

Table 11.1: Notations and characteristic values related to a curve.

11.1.7 Local metric study of arcs in $\mathbb{R}^3$

Let us denote by $(x(\Delta s), y(\Delta s), z(\Delta s))$ the coordinates of $\bar{\gamma}(\Delta s)$ in Frénet's frame of Γ (assumed to be regular in the vicinity of M_0) at the point M_0. The study of the curve's behavior in the vicinity of 0, so for small Δs, leads to looking at a Taylor series at an adequate order in the vicinity of 0. By noticing that order 3 is required to have an analysis that is not restricted to the plane defined by the tangent and the normal, we write the expansion at this order :

$$\bar{\gamma}(\Delta s) = \bar{\gamma}(0) + \Delta s \bar{\gamma}'(0) + \frac{\Delta s^2}{2} \bar{\gamma}''(0) + \frac{\Delta s^3}{6} \bar{\gamma}'''(0) + \mathcal{O}(\Delta s^4),$$

with $\bar{\gamma}(0) = M_0$, $\bar{\gamma}'(0) = \vec{\tau}(0)$, $\bar{\gamma}''(0) = C(0)\vec{\nu}(0)$. Posing M the point $\bar{\gamma}(\Delta s)$, we have :

$$\begin{aligned} M \ &= \ M_0 + \Delta s \vec{\tau}(0) + \tfrac{\Delta s^2}{2} C(0)\vec{\nu}(0) \\ &+ \tfrac{\Delta s^3}{6} \left(-C^2(0)\vec{\tau}(0) + C'(0)\vec{\nu}(0) - C(0)T(0)\vec{b}(0) \right) + \mathcal{O}(\Delta s^4), \end{aligned}$$

hence

$$\left\{ \begin{array}{lcl} x(\Delta s) & = & \Delta s - \frac{C^2(0)\Delta s^3}{6} + \mathcal{O}(\Delta s^4) \\ y(\Delta s) & = & \frac{C(0)\Delta s^2}{2} + \frac{C'(0)\Delta s^3}{6} + \mathcal{O}(\Delta s^4) \\ z(\Delta s) & = & -\frac{C(0)T(0)\Delta s^3}{6} + \mathcal{O}(\Delta s^4) \end{array} \right. \tag{11.15}$$

These relations give the behavior of the projections γ_1, γ_2 and γ_3 of γ on the osculating, rectifying and normal planes to Γ in M_0. To do so, it is sufficient to choose the corresponding plane and the two associated components. For example, to study the behavior of γ_2, we check whether $T(0) < 0$ or whether $T(0) > 0$ (cf. Figure 11.5).

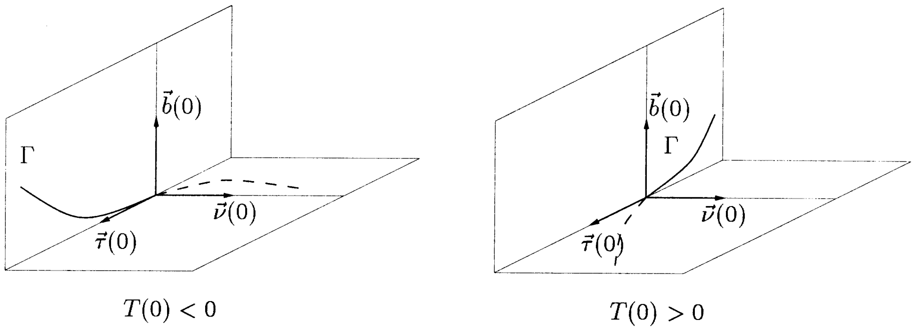

Figure 11.5: *Behavior of the curve γ_2 (projection of Γ on the rectifying plane) according to the sign of the torsion $T(0)$.*

Osculating spheres. Let Γ be a simple arc, which is regular, without any inflection point and of class C^k with $k \geq 3$. Let M_0 be a point of Γ. Then :

Definition 11.4 *There exists an infinity of* osculating spheres *to Γ in M_0, these are all the spheres passing through the osculating circle*[6] *to Γ in M_0.*

Remark 11.6 *The radii of the osculating spheres are greater than or equal to the radius of curvature.*

11.1.8 Parameterization of arcs

In this short section, we are interested in various parameterizations of simple arcs and curves, especially the arcs defined by a Cartesian parameterization and the implicit curves of $\mathbb{R}^2$.

Arcs defined by a Cartesian parameterization. We consider an affine space of finite dimension d.

Definition 11.5 *A* Cartesian parameterization *is such that, in an adequate frame, the parameter is equal to one of the coordinates.*

[6]This is equivalent to saying that such a sphere contains the circle of curvature of γ at this point.

For example, in the affine plane $\mathbb{R}^2$, the parabola defined by :

$$x \longmapsto (x, kx^2), \quad (k = Cte, \ x \in \mathbb{R})$$

is a simple arc.

In a broader sense, in $\mathbb{R}^d$, we consider a parameterization f of the form :

$$x_1 = t, \quad x_2 = f_2(t), \quad \ldots, \quad x_d = f_d(t) \quad (t \in I),$$

the functions f_i $(2 \leq i \leq d)$ being of class C^k on I. An arc of class C^k with $k \geq 1$ admitting such a parameterization is *simple* and *regular*.

Let us consider the parameterization f of the form $(t, f(t))$ in the affine plane $\mathbb{R}^2$. By applying Formula (11.1), we obtain a simplified expression of the length ds of an arc :

$$ds = ||f'(t)||dt = \sqrt{1 + f'^2(t)}\,.$$

Similarly, the tangent vector $\vec{\tau}$ is obtained by the formula :

$$\vec{\tau} = \frac{1}{\sqrt{1 + f'^2(t)}} \begin{pmatrix} 1 \\ f(t) \end{pmatrix},$$

and the (signed) curvature is given by :

$$C = \frac{f''(t)}{(1 + f'^2(t))^{3/2}}\,.$$

Implicit curves in $\mathbb{R}^2$. Let $f : \Omega \longrightarrow \mathbb{R}$ be a function of class C^k $(k \geq 1)$ defined on an open set Ω of the affine plane $\mathbb{R}^2$ and let Γ_f be the set of points $M \in \Omega$ such that $f(M) = 0$.

Definition 11.6 *The pair (f, Γ_f) is the so-called* implicit curve $f = 0$.

Using the implicit function theorem, we can show that the line tangent at M_0 to Γ_0 (arc of class C^k whose local support is the curve Γ_f) is defined by an equation such as :

$$(x - x_0)f'_x(x_0, y_0) + (y - y_0)f'_y(x_0, y_0) = 0\,,$$

where $f(x, y)$ denotes the value of f at the point $M(x, y)$, *i.e.*, we write in the same way both $f(M)$ and $f(x, y)$. Refer to Chapter 16 for more details about implicit curves and functions as well as computing the intrinsic properties of such curves.

We will now move on to the study of the metric properties of surfaces.

11.2 Metric properties of a surface

This section recalls some basic notions useful for the study of surfaces. The aim is once again to give a brief overview of the classical results of differential geometry that can suit our purposes.

In this section, we denote by $\mathcal{E}$ an oriented Euclidean affine space of dimension 3 and by E the associated Euclidean vector space. We recall (by analogy with the curves of E) that a parametric surface [7] of class C^k of $\mathcal{E}$ is an application of class C^k of a domain of $\mathbb{R}^2$ into $\mathcal{E}$.

Let Σ be a regular surface defined by the parameterization σ :

$$\sigma : \Omega \longrightarrow \mathbb{R}^3, \quad (u,v) \longmapsto \sigma(u,v)\,,$$

where Ω denotes a domain of $\mathbb{R}^2$ and σ is a function of class C^k $(k \geq 2)$. By analogy with the study of the local behavior of a curve using a limited expansion, we first indicate what such a development is (at order 2) in the case of a surface.

Let $M = \sigma(u,v)$ be the point of parameters (u,v). We consider the limited expansion at order 2 of σ at the parameters (u,v) for a small increase $(\Delta u, \Delta v)$:

$$\sigma(u+\Delta u, v+\Delta v) = \sigma(u,v) + \underbrace{\sigma'_u \Delta u + \sigma'_v \Delta v}_{\text{order 1}} + \underbrace{\frac{1}{2}\sigma''_{uu}\Delta u^2 + \sigma''_{uv}\Delta u \Delta v + \frac{1}{2}\sigma''_{vv}\Delta v^2}_{\text{order 2}} \tag{11.16}$$

where $\sigma'_u(u,v)$ represents $\frac{\partial \sigma}{\partial u}(u,v)$, $\sigma'_v(u,v)$ denotes $\frac{\partial \sigma}{\partial v}(u,v)$, $\sigma''_{uu}(u,v)$ denotes $\frac{\partial^2 \sigma}{\partial u^2}(u,v)$, $\sigma''_{uv}(u,v)$ denotes $\frac{\partial^2 \sigma}{\partial u \partial v}(u,v)$ and $\sigma''_{vv}(u,v)$ denotes $\frac{\partial^2 \sigma}{\partial v^2}(u,v)$. According to the depth of the limited expansion (at order 1 or 2), we obtain two approximations of the surface allowing to obtain the intrinsic features of the latter. These approximations involve the successive derivatives of σ and, as will be seen, the fundamental forms of the surface Σ.

11.2.1 First fundamental quadratic form

At point $M = M(u,v)$, the tangent vector plane T_M is directed by the vectors :

$$\frac{\partial M}{\partial u} = \sigma'_u(u,v)\,, \qquad \frac{\partial M}{\partial v} = \sigma'_v(u,v)\,,$$

which are respectively noted by $\vec{\tau}_1$ and $\vec{\tau}_2$. Any vector $\vec{V}$ of T_M can be written as :

$$\vec{V} = \lambda\,\vec{\tau}_1 + \mu\,\vec{\tau}_2\,.$$

So, we have :

$$\|\vec{V}\|^2 = \|\lambda\,\vec{\tau}_1 + \mu\,\vec{\tau}_2\|^2 = \lambda^2\|\vec{\tau}_1\|^2 + 2\,\lambda\,\mu\langle\vec{\tau}_1,\vec{\tau}_2\rangle + \mu^2\|\vec{\tau}_2\|^2\,.$$

Posing,

$$E = \|\vec{\tau}_1\|^2\,, \quad F = \langle\vec{\tau}_1,\vec{\tau}_2\rangle \quad \text{and} \quad G = \|\vec{\tau}_2\|^2\,,$$

this expression can be written :

$$\|\vec{V}\|^2 = E\lambda^2 + 2F\lambda\mu + G\mu^2\,.$$

Thus, we have introduced the first fundamental form of the surface, its precise definition being :

[7] Also called a parametric sheet.

Definition 11.7 *Let T_M be the tangent vector plane to a surface Σ at a point M. The restriction to T_M of the quadratic form $\vec{V} \longmapsto \|\vec{V}\|^2$, $(\vec{V} \in E)$, is called the* first fundamental quadratic form *of Σ at M.*

This form is usually denoted by Φ_1^M.

Expression of Φ_1. The usual expression of this form Φ_1^M in the basis $[\vec{\tau}_1, \vec{\tau}_2]$ is then :

$$\Phi_1^M(\vec{V}) = E\lambda^2 + 2F\lambda\mu + G\mu^2\,, \tag{11.17}$$

which can also be written as a "differential" form :

$$\Phi_1^M(dM) = E du^2 \,+\, 2F du dv \,+\, G dv^2\,, \tag{11.18}$$

where $dM = \frac{\partial M}{\partial u}du + \frac{\partial M}{\partial v}dv$.

Posing

$$\Lambda = \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \quad \text{and} \quad \mathcal{M}_1(M) = \begin{pmatrix} E & F \\ F & G \end{pmatrix}\,,$$

we can write this fundamental form using a matrix expression :

$$\Phi_1^M(\vec{V}) \;=\; {}^t\Lambda\,\mathcal{M}_1(M)\,\Lambda\,.$$

Remark 11.7 *The first fundamental form defines the* metric of the tangent plane *to Σ which will be used to govern the calculation of lengths (as will be seen later).*

We then introduce the function $H = \sqrt{EG - F^2}$ allowing[8] us to write :

$$H = \|\vec{\tau}_1 \wedge \vec{\tau}_2\|\,.$$

Case of a Cartesian parameterization. We now assume that Σ is defined by the following Cartesian equation $z = f(x,y)$, where f is a function of class C^k on a planar domain. If $M(x,y)$ denotes the point of coordinates $(x,y,f(x,y))$, we will have the following expressions of E, F and G :

$$E = \left\|\frac{\partial M}{\partial x}\right\|^2 = 1 + f'^2_x(x,y)\,,$$

$$F = \left\langle \frac{\partial M}{\partial x}, \frac{\partial M}{\partial y} \right\rangle = f'_x(x,y) f'_y(x,y)\,,$$

$$G = \left\|\frac{\partial M}{\partial y}\right\|^2 = 1 + f'^2_y(x,y)\,.$$

[8] This function H is notably used to compute the *area* of Σ.

Area of a surface. The quantities E, F and G are used to define a surface element Δa of a surface, corresponding to a small variation Δu and Δv around the point $M(u, v)$. Thus, we have :

$$\Delta a = \left\| \frac{\partial M}{\partial u} \Delta u \wedge \frac{\partial M}{\partial v} \Delta v \right\| ,$$

then :

$$\Delta a = \sqrt{\langle (\vec{\tau}_1 \Delta u \wedge \vec{\tau}_2 \Delta v), (\vec{\tau}_1 \Delta u \wedge \vec{\tau}_2 \Delta v) \rangle} .$$

Hence, using the previously described relations, we reach the expression :

$$\Delta a = \sqrt{EG - F^2} \Delta u \, \Delta v = H \Delta u \, \Delta v .$$

The area of a small element of surface in the vicinity of the point M is then (Figure 11.6) :

$$a = \iint H \Delta u \, \Delta v .$$

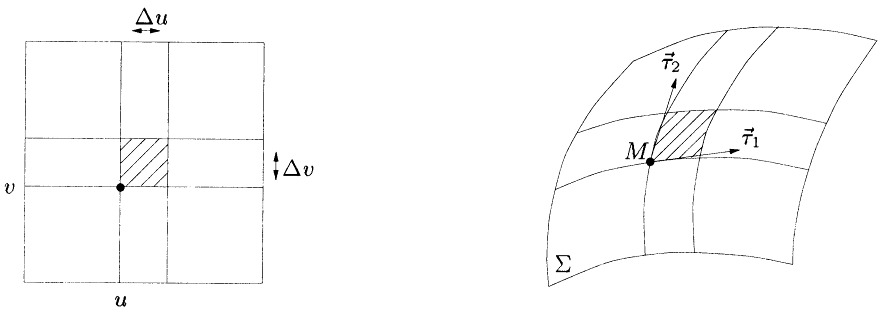

Figure 11.6: *Element of surface defined using (u, v) and $(u + \Delta u, v + \Delta v)$ where Δu (resp. Δv) represents a small increase in u (resp. v).*

We now analyze how to compute the length of an arc traced on a surface. We will see that such a calculation involves the previous fundamental form.

Length of an arc traced on Σ. Let us assume that Γ is the image by the parameterization σ of the planar arc Γ defined by the parameterization : $t \longmapsto (u(t), v(t))$. So, Γ is defined by the parameterization $\sigma : t \longmapsto \sigma(u(t), v(t))$ and thus we have :

$$\sigma' = u'(t) \frac{\partial M}{\partial u} + v'(t) \frac{\partial M}{\partial v} ,$$

allowing us to write :

$$||\sigma'(t)||^2 = Eu'^2(t) + 2Fu'(t)v'(t) + Gv'(t) .$$

Indeed, if $[a, b]$ is an interval, the length of Γ is :

$$L(\Gamma) = \int_a^b \sqrt{Eu'^2 + 2Fu'v' + Gv'^2} dt ,$$

and finally (posing $du = u'(t)$ and $dv = v'(t)$) :

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2 ,$$

or also

$$ds = \sqrt{Edu^2 + 2Fdudv + Gdv^2} .$$

Remark 11.8 *We can also express the length of an arc traced on Σ with respect to t :*

$$s(t) = \int_a^b \sqrt{Edu^2 + 2Fdudv + Gdv^2} .$$

11.2.2 Normal. Local frame. Darboux's frame

We will then focus on the notion of normal and oriented normal. Let Σ be a simple surface of class C^k $(k \geq 1)$ of $\mathcal{E}$ defined by the parameterization $\sigma : (u, v) \longrightarrow M(u, v) = \sigma(u, v)$.

Normal. We pose :

$$\vec{N}(u, v) = \vec{\tau}_1 \wedge \vec{\tau}_2 ,$$

with $\vec{\tau}_1$ and $\vec{\tau}_2$ the vectors of the basis of the tangent plane previously introduced. With the notations introduced before, we can write :

$$||\vec{N}(u, v)|| = H(u, v) \quad \text{with} \quad H = \sqrt{EG - F^2} .$$

The vector $\vec{N}(u, v)$ thus defined is called the *normal vector* to Σ at M associated with the given parameterization σ. The affine line passing through M and directed by the vector $\vec{N}$ (thus orthogonal to the tangent plane) is called the *normal* to Σ at M.

Oriented normal. The unit vector

$$\vec{n}(u, v) = \frac{\vec{N}(u, v)}{H(u, v)} = \frac{\vec{\tau}_1 \wedge \vec{\tau}_2}{||\vec{\tau}_1 \wedge \vec{\tau}_2||}$$

does not depend on the orientation of Σ with respect to the given parameterization. This vector is the *unit normal vector* at M to Σ.

Local frame. At the point $M = \sigma(u,v)$, the tangent plane T_M can be defined by means of the parametric equation : $M + \lambda \vec{\tau}_1 + \mu \vec{\tau}_2$. The normal $\vec{\tau}_1 \wedge \vec{\tau}_2$ to the plane T_M coincides with the normal to Σ at M. So, the unit normal

$$\vec{n} = \frac{\vec{\tau}_1 \wedge \vec{\tau}_2}{||\vec{\tau}_1 \wedge \vec{\tau}_2||}$$

and the vectors $\vec{\tau}_1$ and $\vec{\tau}_2$ form a local system of coordinates :

$$\mathcal{F}_{loc} = [M, \vec{\tau}_1, \vec{\tau}_2, \vec{n}]\,,$$

the so-called *local frame* at M (Figure 11.7, left-hand side).

Remark 11.9 *Notice also that the axes $\vec{\tau}_1$ and $\vec{\tau}_2$ usually form only an affine system. This frame is the analogue of the Frénet frame for the curves.*

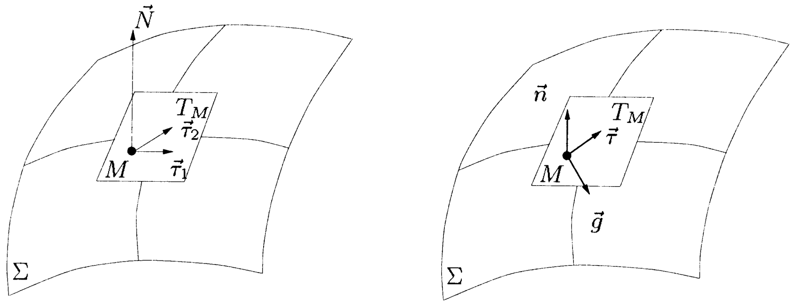

Figure 11.7: *Left-hand side : the local frame at point $M = \sigma(u,v)$ of Σ. Right-hand side : the Darboux frame (moving frame) associated with point $M = \sigma(u,v)$ of Σ and with the tangent $\vec{\tau}$.*

Darboux's frame. The *Darboux frame* $[M, \vec{\tau}, \vec{g}, \vec{n}]$ associated with the pair $(M, \vec{\tau})$ is defined by $\vec{\tau}$, a unit tangent vector at M to Σ, by $\vec{n}$, the unit normal vector and by the vector $\vec{g} = \vec{n} \wedge \vec{\tau}$ (Figure 11.7, right-hand side).

We now consider a regular and oriented arc Γ traced on Σ. The moving frame

$$\mathcal{F}_{moving} = [\gamma(s), \vec{\tau}(s), \vec{g}(s), \vec{n}(s)]$$

is the *Darboux frame* of Γ (associated with a normal parameterization $s \longmapsto \gamma(s)$). Moreover, the vector $\vec{g}(s)$ is called the *geodesic normal vector* to Γ at the point $M = \gamma(s)$.

Given a curve Γ traced on Σ, the tangent $\vec{\tau}$ to Γ at M is one of the vectors of T_M and a moving frame can be associated with this particular tangent vector.

The study of the various curves passing through a point M allows us to capture the behavior of the surface in the vicinity of M. For instance, the intersection of Σ with any of the planes containing the unit normal $\vec{n}$ defines one of the curves we are interested in. Such a plane Π_n is a *normal section* to the surface at M.

11.2.3 Normal curvature, curvature and geodesic torsion

The *normal curvature* of Γ is defined by the function :

$$\kappa_n : s \longmapsto \kappa_n(s) = \left\langle \vec{n}, \frac{d\vec{\tau}}{ds} \right\rangle = -\left\langle \vec{\tau}, \frac{d\vec{n}}{ds} \right\rangle .$$

The *geodesic curvature* of Γ is defined by the function :

$$\kappa_g : s \longmapsto \kappa_g(s) = \left\langle \vec{g}, \frac{d\vec{\tau}}{ds} \right\rangle = -\left\langle \vec{\tau}, \frac{d\vec{g}}{ds} \right\rangle .$$

Finally, the *geodesic torsion* of Γ is defined by the function :

$$T_g : s \longmapsto T_g(s) = \left\langle \vec{g}, \frac{d\vec{n}}{ds} \right\rangle = -\left\langle \vec{n}, \frac{d\vec{g}}{ds} \right\rangle .$$

Using all these definitions, the *Darboux formulas* are the following :

$$\frac{d\vec{\tau}}{ds} = \kappa_g \vec{g} + \kappa_n \vec{n}\,, \quad \frac{d\vec{g}}{ds} = -\kappa_g \vec{\tau} - T_g \vec{n}\,, \quad \frac{d\vec{n}}{ds} = -\kappa_n \vec{\tau} + T_g \vec{g}\,.$$

11.2.4 Second fundamental form

At the point $M = M(u,v)$, any vector $\vec{\tau}$ of the tangent plane T_M can be written as $\vec{\tau} = \lambda\,\vec{\tau}_1 + \mu\,\vec{\tau}_2$. Let $\vec{n}$ be the unit normal vector to T_M at M defined above. Using the same notations as in the previous section, we have :

$$\kappa_n = \left\langle \vec{n}, \frac{d\vec{\tau}}{ds} \right\rangle = -\left\langle \vec{\tau}, \frac{d\vec{n}}{ds} \right\rangle$$

where

$$\vec{n}(u,v) = \frac{\vec{N}(u,v)}{H(u,v)}\,.$$

Developing the first formula, we obtain a relationship of the following form :

$$\kappa_n = L\left(\frac{du}{ds}\right)^2 + 2M\frac{du}{ds}\frac{dv}{ds} + N\left(\frac{dv}{ds}\right)^2, \tag{11.19}$$

where :

$$L = \left\langle \vec{n}, \frac{\partial^2 M}{\partial u^2} \right\rangle \qquad M = \left\langle \vec{n}, \frac{\partial^2 M}{\partial u \partial v} \right\rangle \qquad N = \left\langle \vec{n}, \frac{\partial^2 M}{\partial v^2} \right\rangle .$$

Formula (11.19) indicates that the normal curvature of Γ at M depends only on the direction of the unit tangent vector :

$$\vec{\tau} = \frac{\partial M}{\partial u}\frac{du}{ds} + \frac{\partial M}{\partial v}\frac{dv}{ds}\,.$$

Thus we introduce the second fundamental form of the surface :

Definition 11.8 *Let T_M be the tangent vector plane to the surface Σ at M, the* second fundamental quadratic form *of Σ at M is the quadratic form :*

$$\Phi_2^M(\vec{V}) = L\lambda^2 + 2M\lambda\mu + N\mu^2\,,$$

the values L, M, N being given by the formulas of Relation (11.19).

Expression of Φ_2. The expression of Φ_2^M can also be written as a differential form :

$$\Phi_2^M(dM) = Ldu^2 + 2Mdudv + Ndv^2\,,$$

with $dM = \frac{\partial M}{\partial u}du + \frac{\partial M}{\partial v}dv$. Posing

$$\mathcal{M}_2(M) = \begin{pmatrix} L & M \\ M & N \end{pmatrix},$$

we can write Φ_2^M as a matrix form (as we did for Φ_1^M) :

$$\Phi_2^M(\vec{V}) = {}^t\Lambda\,\mathcal{M}_2(M)\,\Lambda\,.$$

Remark 11.10 *The second fundamental form measures the deviation between the tangent plane and the surface at M.*

Case of a Cartesian parameterization. Let us assume that the surface Σ is parameterized by the Cartesian equation $z = \sigma(x,y)$, where σ is of class C^2 at least. Let $M(x,y)$ be the point of coordinates $(x, y, \sigma(x,y))$, the second fundamental form is :

$$\Phi_2(dM) = \frac{1}{H}(rdx^2 + 2sdxdy + tdy^2)\,,$$

with

$$dM = \frac{\partial M}{\partial x}dx + \frac{\partial M}{\partial y}dy\,,$$

and

$$H = \sqrt{1 + p^2 + q^2}\,,$$

posing : $p = \sigma'_x$, $q = \sigma'_y$, $r = \sigma''_{x^2}$, $s = \sigma''_{xy}$ and $t = \sigma''_{y^2}$.

11.2.5 Computing curvatures and geodesic torsion

The normal curvature κ_n at point M of Σ is related to a tangent vector of the form $\vec{V} = \lambda\vec{\tau}_1 + \mu\vec{\tau}_2$ by the relation :

$$\kappa_n(\vec{V}) = \frac{\Phi_2^M(\vec{V})}{\Phi_1^M(\vec{V})}\,.$$

Whenever $\vec{V}$ varies, this function admits two extrema, the *principal curvatures* to which are associated the two *principal radii of curvature* and the two *principal directions*.

Remark 11.11 *We consider the curve traced on the surface and defined by the segment joining (u,v) to $(u+\lambda, v+\mu)$. We have seen previously that this curve could be approached by a parabola with a limited expansion truncated at order 2. Using Pythagorus's theorem, we can then write :*

$$\left(R(\vec{V}) - \frac{1}{2}\Phi_2^M(\vec{V})\right)^2 + \left(\Phi_1^M(\vec{V})\right) = R(\vec{V})^2\,,$$

and, neglecting the terms at order 2 before those at order 1, we have :

$$R(\vec{V}) = \left(\frac{\Phi_1^M(\vec{V})}{\Phi_2^M(\vec{V})}\right).$$

All this information can be used to control the quality of the geometric polyhedral approximation of the surface (cf. Chapter 15).

Curvatures. Let κ_1 be the minimal curvature and let κ_2 be the maximal curvature. We define the *Gauss curvature*[9], the *mean curvature* and the *absolute curvature* respectively as :

$$\kappa_{Gauss} = \kappa_1\kappa_2\,,$$

$$\kappa_{Mean} = \frac{\kappa_1 + \kappa_2}{2}\,,$$

$$\kappa_{Abs} = |\kappa_1| + |\kappa_2|\,.$$

Geodesic torsion. Let Γ be an arc defined by a normal parameterization and traced on Σ. The geodesic torsion of Γ is given by the formula :

$$T_g = \frac{1}{H}\left\langle \frac{\partial M}{\partial u}\wedge\frac{\partial M}{\partial v}\,,\ \vec{\tau}\wedge\frac{d\vec{n}}{ds}\right\rangle,$$

which can also be written as :

$$T_g = \frac{1}{H}\begin{vmatrix} \left\langle \vec{\tau}, \frac{\partial M}{\partial u}\right\rangle & \left\langle \vec{\tau}, \frac{\partial M}{\partial v}\right\rangle \\ \left\langle \frac{d\vec{n}}{ds}, \frac{\partial M}{\partial u}\right\rangle & \left\langle \frac{d\vec{n}}{ds}, \frac{\partial M}{\partial v}\right\rangle \end{vmatrix}.$$

11.2.6 Meusnier's circle and theorem

We assume that the normal curvature κ_n is not null and we denote by $\Omega_n = M + R_n\vec{n}$ the center of normal curvature. If Γ is a simple and regular arc traced on Σ and tangent to Σ at M, the center of curvature Ω of Γ at M is defined by :

$$\Omega = M + R\vec{\nu}\,, \qquad \text{with} \quad R = \frac{1}{C} = \frac{\cos\alpha}{\kappa_n} = R_n\cos\alpha\,.$$

where α denotes the angle between the normal $\vec{\nu}$ to Γ and the normal $\vec{n}$ to Σ at M. The point Ω is the projection of Ω_n onto the principal normal to γ at M.

The relation $\kappa_n(\vec{\tau}) = Ccos\alpha$ has an interesting application. If Γ is the intersection of Σ by a normal section, we find that the circle of diameter $\kappa_n(\vec{\tau})$, the *Meusnier circle*, is the geometric locus of points M, endpoints of the segments PM, such that $||PM|| = C$, the curvature of a curve Γ whose normal $\vec{\nu}$ makes an angle α with the unit normal $\vec{n}$ to Σ. In other words, $\kappa_n(\vec{\tau})$ and α make it possible to obtain the curvature of various curves under this angle.

[9] Also called the *total curvature*.

11.2.7 Local behavior of a surface

In the local study of curves (Section 11.1.4), we have seen that the curvature gives an indication of the deviation of the curve with respect to the tangent in the vicinity of a point M. Similarly, we will analyze the behavior of a surface with respect to its tangent planes.

Let Σ be a simple and regular surface of class at least C^2, defined by a parameterization σ. We wish study the behavior of the surface Σ in the neighborhood of a point $M = \sigma(u, v)$.

We consider a small increase $\Delta M = (\Delta u, \Delta v)$ in u and v in the neighborhood of M, assumed to be sufficiently small. Applying a Taylor series at order 1 to σ, we have :

$$\sigma(u+\Delta u, v+\Delta v) = \sigma(u,v) + \Delta u\,\frac{\partial M}{\partial u} + \Delta v\,\frac{\partial M}{\partial v} + \mathcal{O}(\|\Delta M\|^2)\,.$$

This local study corresponds to an approximation of the surface by a plane (the tangent plane), noted T_M, directed by the vectors $\vec{\tau}_1$ and $\vec{\tau}_2$ at point M.

To estimate the gap between the surface and the approximation, we pursue the development at order 2 (for a surface smooth enough) :

$$\begin{aligned}\sigma(u+\Delta u, v+\Delta v) &= \sigma(u,v) + \Delta u\,\vec{\tau}_1 + \Delta v\,\vec{\tau}_2\\ &+\frac{1}{2}\left(\Delta u^2\,\frac{\partial^2 M}{\partial u^2} + 2\,\Delta u\,\Delta v\,\frac{\partial^2 M}{\partial u\,\partial v} + \Delta v^2\,\frac{\partial^2 M}{\partial v^2}\right) + \mathcal{O}(\|\Delta M\|^3)\,.\end{aligned}$$

In this expression, the term of order 2 enables us to measure the desired gap. The local analysis corresponds to an approximation of the surface by a paraboloid (a quadric) $\mathcal{P}$. Notice that in this development, we can exhibit the terms L, M and N previously introduced by observing the projection on $\vec{n}$, we have :

$$\langle\Delta u^2\,\frac{\partial^2 M}{\partial u^2} + 2\,\Delta u\,\Delta v\,\frac{\partial^2 M}{\partial u\,\partial v} + \Delta v^2\,\frac{\partial^2 M}{\partial v^2}\,,\,\vec{n}\rangle \;=\; \Delta u^2\,L + 2\Delta u\Delta v\,M + \Delta v^2\,N\,,$$

that makes use of the second fundamental form Φ_2 of the surface (*i.e.*, the quadratic form associated to σ'' at M). The nature of the form Φ_2 corresponds to a local geometric property of Σ at the point M.

In the Taylor series at order 2, the second order term allows us to determine whether the tangent plane T_M intersects the surface in the neighborhood of M.

Dupin's indicatrix. Using the polar coordinates, $r = \sqrt{\rho}$ and θ, a point P of the tangent plane T_M at M can be written as :

$$x = \sqrt{\rho}\cos\theta \qquad \text{and} \qquad y = \sqrt{\rho}\sin\theta\,,$$

where θ denotes the angle between one of the principal directions (the other being orthogonal) and the line MP. We can then use the Euler relation [doCarmo-1976] :

$$\kappa_n = \kappa_1\cos^2\theta + \kappa_2\sin^2\theta\,,$$

relating the minimal and maximal curvatures to the normal curvature and, then, we can write :

$$\kappa_1\,x^2 + \kappa_2\,y^2 = \frac{x^2}{\rho_1} + \frac{y^2}{\rho_2} = \pm 1\,. \tag{11.20}$$

which corresponds to the equation of a conic section, called the *Dupin indicatrix*. Notice, by the way, that a change in the orientation of the normal does not change the sign of ρ, thus the symbol $\pm$ in the relation.

Let us now consider a plane T_ε, parallel to the tangent plane T_M at M and located at a distance ε of T_M. Considering the Taylor series at order 2 of σ, the trace of the quadric in this plane is such that :

$$\langle\Delta u^2\,\frac{\partial^2 M}{\partial u^2} + 2\,\Delta u\,\Delta v\,\frac{\partial M}{\partial u\,\partial v} + \Delta v^2\,\frac{\partial^2 M}{\partial v^2}\,,\,\vec{n}\rangle \;=\; \varepsilon\,,$$

The idea is here to consider that the Dupin indicatrix represents the intersection of the surface (locally approached by the paraboloid) and the plane T_ε (within a scale factor ζ, such that $\varepsilon = \frac{1}{2\zeta^2}$).

We study the behavior of the curve defined by Equation (11.20) in the plane T_ε, according to the curvatures κ_1 and κ_2, by noticing that the conic is a curve not depending on the parameterization of Σ. This curve is (Figure 11.8) :

- an ellipse, when $\kappa_1\,\kappa_2 > 0$,
- two parallel lines (at a distance of $2\rho_1$ or $2\rho_2$), when $\kappa_1 = 0$ or $\kappa_2 = 0$,
- a pair of hyperbolas, when $\kappa_1\,\kappa_2 < 0$.

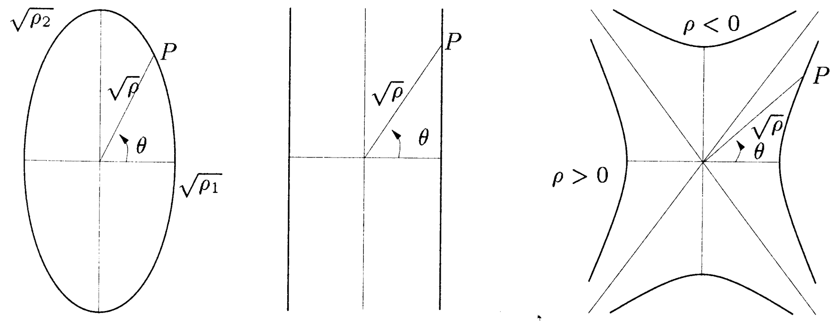

Figure 11.8: *Dupin's indicatrix : elliptical point, $\kappa_1\,\kappa_2 > 0$ (left-hand side), parabolic point, $\kappa_1 = 0$ or $\kappa_2 = 0$ (middle) and hyperbolic point, $\kappa_1\,\kappa_2 < 0$ (right-hand side).*

From a geometric point of view, when the distance ε between T_ε and T_M tends towards 0, the curve is reduced down to a contact point only. When decreasing the distance, the curve is scaled up continuously, we observe that this curve approaches an ellipsis (in the case of an elliptical point) whose center is the contact point.

11.3 Computational issues about surfaces

In this last section, we will focus on problems which commonly arise when dealing with surfaces.

11.3.1 Curvature computation

Let Σ be a simple regular and oriented surface of class C^k $(k \geq 2)$ defined by a parameterization $(u,v) \longmapsto M(u,v)$. Consider a curve Γ traced in a normal section Π_n at a point M to the surface.

The curvature of Γ at a point M characterizes the geometry of the curve in the neighborhood of this point. The *normal curvature* characterizes the surface in the neighborhood of the same point. The expression of the curvature of Γ shows the tangent vector $\vec{\tau}$ and the unit normal $\vec{\nu}$ as follows :

$$\frac{d\vec{\tau}(t)}{ds} = C\vec{\nu}(t) \quad \text{or} \quad C = \left\| \frac{d^2\gamma(t)}{ds^2} \right\| ,$$

whereas the normal curvature of the surface at M is characterized by :

$$\kappa = \kappa_n(\vec{\tau}) = \left\langle \frac{d\vec{\tau}(s)}{ds}, \vec{n} \right\rangle = \left\langle \frac{d^2\gamma(s)}{ds^2}, \vec{n} \right\rangle .$$

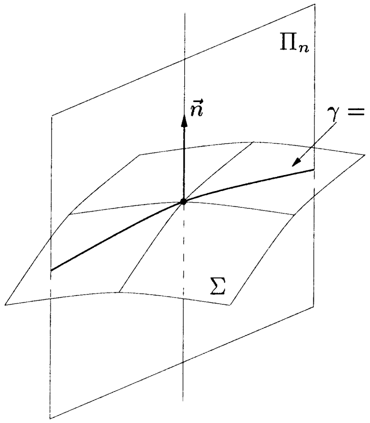

Figure 11.9: *A normal section Π_n and the curve Γ representing the intersection of the surface with this plane.*

In fact, $\kappa_n(\vec{\tau})$ measures the curvatures of the surface with respect to its unit normal $\vec{n}$ at M in the direction $\vec{\tau}$. Thus, we have :

$$\kappa_n(\vec{\tau}) = C \cos\alpha ,$$

where α is the angle between $\vec{\nu}$ and $\vec{n}$.

Remark 11.12 *If $\vec{n} = \vec{\nu}$ (when these two vectors are colinear), the curve Γ is called a* geodesic. *In other words, at point M, the unit normal $\vec{n}$ to Σ is the* geodesic normal *of γ.*

As we assumed a parameterization σ of Σ, the relations

$$\gamma(t) = \sigma(u(t), v(t)) \quad \text{or} \quad \gamma(s) = \sigma(u(s), v(s)) ,$$

are established, the characteristics of Σ (*i.e.*, its normal curvatures) can be obtained with respect to σ and to its derivatives.

As $\langle \vec{n}, \vec{\tau} \rangle = 0$,

$$\kappa = \left\langle \frac{d\vec{\tau}(s)}{ds}, \vec{n} \right\rangle \quad \text{is also} \quad \kappa = -\left\langle \vec{\tau}, \frac{d\vec{n}}{ds} \right\rangle .$$

Denoting $\vec{n} = \frac{\vec{\tau}_1 \wedge \vec{\tau}_2}{H}$, we have $H\vec{n} = \vec{\tau}_1 \wedge \vec{\tau}_2$ and using again $\langle \vec{n}, \vec{\tau} \rangle = 0$,

$$\left\langle \vec{\tau}, \frac{dH\vec{n}}{ds} \right\rangle = \left\langle \vec{\tau}, \frac{dH}{ds}\vec{n} \right\rangle + H \left\langle \vec{\tau}, \frac{d\vec{n}}{ds} \right\rangle$$

is finally reduced to :

$$\left\langle \vec{\tau}, \frac{dH\vec{n}}{ds} \right\rangle = H \left\langle \vec{\tau}, \frac{d\vec{n}}{ds} \right\rangle ,$$

hence

$$\kappa = -\left\langle \vec{\tau}, \frac{d\vec{n}}{ds} \right\rangle = -\frac{1}{H} \left\langle \vec{\tau}, \frac{d(\vec{\tau}_1 \wedge \vec{\tau}_2)}{ds} \right\rangle .$$

Now, depending on u and v, as

$$\frac{d(\vec{\tau}_1 \wedge \vec{\tau}_2)}{ds} = \left(\frac{d\vec{\tau}_1}{ds} \wedge \vec{\tau}_2) + (\vec{\tau}_1 \wedge \frac{d\vec{\tau}_2}{ds} \right)$$

with

$$\frac{d\vec{\tau}_1}{ds} = \frac{\partial \vec{\tau}_1}{\partial u}\frac{du}{ds} + \frac{\partial \vec{\tau}_1}{\partial v}\frac{dv}{ds} \quad \text{and} \quad \frac{d\vec{\tau}_2}{ds} = \frac{\partial \vec{\tau}_2}{\partial u}\frac{du}{ds} + \frac{\partial \vec{\tau}_2}{\partial v}\frac{dv}{ds}$$

we thus have :

$$\frac{d(\vec{\tau}_1 \wedge \vec{\tau}_2)}{ds} = \left(\left(\frac{\partial \vec{\tau}_1}{\partial u}\frac{du}{ds} + \frac{\partial \vec{\tau}_1}{\partial v}\frac{dv}{ds} \right) \wedge \vec{\tau}_2 \right) + \left(\vec{\tau}_1 \wedge \left(\frac{\partial \vec{\tau}_2}{\partial u}\frac{du}{ds} + \frac{\partial \vec{\tau}_2}{\partial v}\frac{dv}{ds} \right) \right) .$$

Finally, the expression of the curvature κ_n) is given by the following formula :

$$-\frac{1}{H} \left\langle \left(\vec{\tau}_1\frac{du}{ds} + \vec{\tau}_2\frac{dv}{ds} \right), \left(\left(\frac{\partial \vec{\tau}_1}{\partial u}\frac{du}{ds} + \frac{\partial \vec{\tau}_1}{\partial v}\frac{dv}{ds} \right) \wedge \vec{\tau}_2 \right) + \left(\vec{\tau}_1 \wedge \left(\frac{\partial \vec{\tau}_2}{\partial u}\frac{du}{ds} + \frac{\partial \vec{\tau}_2}{\partial v}\frac{dv}{ds} \right) \right) \right\rangle ,$$

then, for instance L, the coefficient depending on $(du/ds)^2$ is :

$$L = -\frac{1}{H} \left\langle \vec{\tau}_1, \left(\frac{\partial \vec{\tau}_1}{\partial u} \wedge \vec{\tau}_2 + \vec{\tau}_1 \wedge \frac{\partial \vec{\tau}_2}{\partial u} \right) \right\rangle = -\frac{1}{H} \langle \vec{\tau}_1, (\sigma''_{uu} \wedge \vec{\tau}_2) \rangle = \frac{1}{H} \langle \vec{\tau}_1 \wedge \vec{\tau}_2, \sigma''_{uu} \rangle$$

which is simply :

$$\left\langle \frac{\vec{\tau}_1 \wedge \vec{\tau}_2}{H}, \sigma''_{uu} \right\rangle \quad i.e., \quad \langle \vec{n}, \sigma''_{uu} \rangle ,$$

previously defined. Similarly, we can demonstrate that :

$$M = \langle \vec{n}, \sigma''_{uv} \rangle \quad \text{and} \quad N = \langle \vec{n}, \sigma''_{vv} \rangle .$$

Exercise 11.1 *Establish the previous results for M and N (Hint : find the terms in $du/ds\, dv/ds$ and $(dv/ds)^2$, respectively).*

Using these values, the normal curvature is given by :

$$\kappa = L\left(\frac{du}{ds}\right)^2 + 2\,M\frac{du}{ds}\frac{dv}{ds} + N\left(\frac{dv}{ds}\right)^2 \quad \text{or} \quad \kappa = \frac{Ldu^2 + 2\,Mdudv + Ndv^2}{ds^2}$$

i.e.,

$$\kappa_n(\vec{\tau}) = \kappa = \frac{Ldu^2 + 2\,Mdudv + Ndv^2}{Edu^2 + 2\,Fdudv + Gdv^2} = \frac{\Phi_2^M(\vec{\tau})}{\Phi_1^M(\vec{\tau})} \qquad (11.21)$$

according to a given direction $\vec{\tau}$ such that $\vec{\tau} = du\vec{\tau}_1 + dv\vec{\tau}_2$. Thus, we find the relation between the two fundamental forms (evaluated in $\vec{\tau}$) introduced previously.

11.3.2 Normal curvature analysis

Let λ and μ be two parameters and consider $\vec{\tau} = \lambda\vec{\tau}_1 + \mu\vec{\tau}_2$. Using the previously established result, we can write :

$$\kappa_n(\vec{\tau}) = \kappa_n(\lambda, \mu) = \frac{L\lambda^2 + 2\,M\lambda\mu + N\mu^2}{E\lambda^2 + 2\,F\lambda\mu + G\mu^2} ,$$

the aim is to determine how this function varies. To this end, we study its variation with respect to the two parameters. We start by searching the extrema of this function, which are given by the relations :

$$\frac{\partial\kappa_n(\lambda,\mu)}{\partial\lambda} = 0 \quad \text{and} \quad \frac{\partial\kappa_n(\lambda,\mu)}{\partial\mu} = 0 .$$

These two relations lead to a unique equation :

$$(FL - EM)\lambda^2 + (GL - EN)\lambda\mu + (GM - FN)\mu^2 = 0 . \qquad (11.22)$$

Notice that this relation can also be written in the following form :

$$det \begin{vmatrix} \mu^2 & -\lambda\mu & \lambda^2 \\ E & F & G \\ L & M & N \end{vmatrix} = 0 . \qquad (11.23)$$

If the three coefficients of the relations are null, then the two fundamental forms are proportional, irrespective of the values of $\vec{\tau}$ (*i.e.*, for all (λ, μ)) and the curvature is constant along the normal sections. In this case, the constant curvature is then :

$$\kappa_n(\vec{\tau}) = \kappa_n = \frac{L}{E} = \frac{M}{F} = \frac{N}{G} .$$

In the other cases, this equation admits two distinct solutions (*i.e.*, two pairs (λ_1, μ_1) and (λ_2, μ_2)). With each of these pairs is associated a vector, respectively :

$$\vec{V}_1 = \lambda_1\vec{\tau}_1 + \mu_1\vec{\tau}_2 \quad \text{and} \quad \vec{V}_2 = \lambda_2\vec{\tau}_1 + \mu_2\vec{\tau}_2 .$$

$\vec{V}_1$ and $\vec{V}_2$ are orthogonal, *i.e.*, $\langle \vec{V}_1, \vec{V}_2 \rangle = 0$. They define the *two principal directions*. The vectors

$$\vec{W}_1 = \frac{\vec{V}_1}{||\vec{V}_1||} \quad \text{and} \quad \vec{W}_2 = \frac{\vec{V}_2}{||\vec{V}_2||}$$

form with $\vec{n}$ an orthonormal basis called the *local principal basis* at point M and denoted as :

$$\mathcal{B}_M = [M, \vec{W}_1, \vec{W}_2, \vec{n}] .$$

With each principal direction is associated a *principal curvature*, which is the solution of :

$$det \begin{vmatrix} \kappa E - L & \kappa F - M \\ \kappa F - M & \kappa G - N \end{vmatrix} = 0 . \qquad (11.24)$$

To summarize, the surface Σ at point M is characterized by its local principal basis $\mathcal{B}_M$, its two principal curvatures $\kappa_1 = \kappa_n(\lambda_1, \mu_1)$ and $\kappa_2 = \kappa_n(\lambda_2, \mu_2)$ and thus by its two principal radii of curvature $\rho_1 = 1/\kappa_1$ and $\rho_2 = 1/\kappa_2$ whose variations allow us to determine locally the geometry of the surface.

Remark 11.13 *By comparing this result with the expression of the geodesic torsion defined previously, we again find the fact that the two principal directions are the directions for which the geodesic torsion is null.*

Remark 11.14 *All the functions introduced in this section will serve to define the theoretical requisites needed to control the meshing process of surfaces (cf. Chapter 15).*

11.3.3 Relationship between curvatures and fundamental forms

We suggest here a different way of establishing the results given above. To this end, let us consider the two metrics associated with the two fundamental forms. According to the previous sections, we already know that :

$$\mathcal{M}_1(M) = \begin{pmatrix} E & F \\ F & G \end{pmatrix} ,$$

is attached to the first fundamental form and that :

$$\mathcal{M}_2(M) = \begin{pmatrix} L & M \\ M & N \end{pmatrix} ,$$

corresponds to the second fundamental form.

Let us consider the matrix $\mathcal{N} = \mathcal{M}_1^{-1}\mathcal{M}_2$. This matrix is diagonalizable (as it is a $\mathcal{M}_1$-symmetric matrix). Let $\vec{W}_1$ and $\vec{W}_2$ be the two unit eigenvectors of $\mathcal{N}$. Then, for each vector $\vec{V}$ of T_M, expressed according to $\vec{W}_1$ and $\vec{W}_2$, we have :

$$\Phi_1(\vec{V}) = \alpha^2 + \beta^2 \quad \text{and} \quad \Phi_2(\vec{V}) = \kappa_1\alpha^2 + \kappa_2\beta^2$$

leading to :

$$\kappa_n(\vec{V}) = \frac{\Phi_2(\vec{V})}{\Phi_1(\vec{V})} = \frac{\kappa_1\alpha^2 + \kappa_2\beta^2}{\alpha^2 + \beta^2} .$$

The extrema of $\kappa_n(\vec{V})$ are obtained for $\alpha = 0$ or $\beta = 0$. Hence, κ_1 and κ_2 are naturally extrema (in the previous expression).

The corresponding directions (*i.e.*, the $\vec{V}$'s) are $\vec{W}_1$ (for $\beta = 0$) and $\vec{W}_2$ (for $\alpha = 0$).

Remark 11.15 *In fact, finding the principal directions is only a geometric interpretation of the problem of the simultaneous reduction of two fundamental quadratic forms (cf. Chapter 10).*

11.3.4 Local behavior of a surface

The analysis of the local principal basis when M varies on Σ allows us to capture the local behavior of the surface. Indeed, the extremal values of κ, the curvature related to a point, characterize the type of the surface at point M. We can also write :

$$\kappa^2 - (\kappa_1 + \kappa_2)\kappa + \kappa_1\kappa_2 = 0 ,$$

or

$$\kappa^2 - 2\,\kappa_{Mean}\kappa + \kappa_{Gauss} = 0 . \qquad (11.25)$$

Thus we have :

$$\kappa_{Mean} = \frac{1}{2}\frac{NE - 2MF + LG}{EG - F^2} \quad \text{and} \quad \kappa_{Gauss} = \frac{LN - M^2}{EG - F^2} .$$

When (cf. Figure 11.10) :

- $\kappa_{Gauss} > 0$, the point M is said to be *elliptic*,
- $\kappa_{Gauss} < 0$, the point M is said to be *hyperbolic*,
- $\kappa_{Gauss} = 0$, the point M is said to be *parabolic*.

Obviously, when κ_{Gauss} and κ_{Mean} both tend towards 0, the surface is *planar* at M.

Remark 11.16 *For a point M of a simple and regular surface to be* elliptic, *it is necessary and sufficient that the second fundamental form is defined (positive or negative). For the point M to be* hyperbolic, *it is necessary and sufficient that the second fundamental form Φ_2^M is non degenerated and non defined. Finally, for the point M to be* parabolic, *it is necessary and sufficient that the second form Φ_2^M is degenerated.*

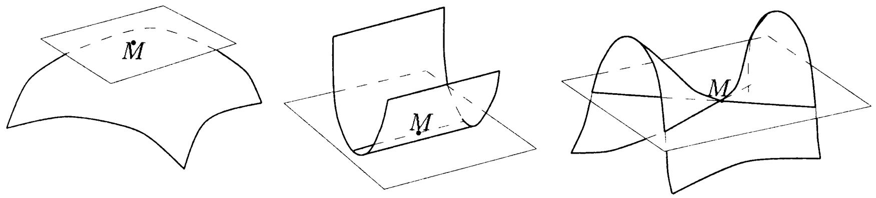

Figure 11.10: *Position of a surface with respect to a tangent plane. Elliptic point (left-hand side), parabolic point (middle) and hyperbolic point (right-hand side).*

11.4 Non-linear problems

The definitions of curves and surfaces usually leave out some problems. For instance, the intersection procedures between two curves, two pieces of surfaces (patches such those defined in Chapter 13) theoretically lead to solving a non-linear equation. In the general case, these problems can also be solved explicitly and therefore require the development of numerical approximation methods. Some of these methods are presented in this section.

11.4.1 Non-linear issues

Let Γ_1 and Γ_2 be two parametric curves associated with the functions $\gamma_1(t)$ and $\gamma_2(t)$. The intersection of these curves is defined by the set of pairs of values :

$$(t_1, t_2) \text{ such that } \gamma_1(t_1) = \gamma_2(t_2)$$

leading to a non-linear equation. Similarly let us consider Σ_1 and Σ_2 two bi-parametric patches, associated with the functions $\sigma_1(u_1, v_1)$ and $\sigma_2(u_2, v_2)$. The intersection between Σ_1 and Σ_2 is defined by the set of pairs of vectors :

$$(u_1, v_1, u_2, v_2) \text{ such that } \sigma_1(u_1, v_1) = \sigma_2(u_2, v_2)$$

leading also to a non-linear equation. Finally, let us consider the intersection between a patch and a curve, we then have the following (non-linear) equation to solve :

$$(u_1, v_1, t) \quad \text{such that} \quad \sigma_1(u_1, v_1) = \gamma_1(t_1) .$$

Moreover, the projection and the search for the closest point on such a curve from a given point according to a specific direction leads to a non-linear equation of the following type :

$$t \text{ such that } ||M\gamma(t)|| \text{ is minimum, or also } \frac{d||M, \gamma(t)||}{dt} = 0 .$$

In such cases, the problems can be solved with the algorithms described in the following sections. However, we should also mention here the algebraic methods. This type of method is usually based on an implicit definition of $\vec{F}$ ($\vec{F}$ being a vector with d components representing the problem to be solved), $\vec{F} = 0$. More details about such methods can be found in the literature [Sederberg-1987]. We will not spend more time here on these issues as they are seldom used in this context.

11.4.2 Newton-Raphson type algorithms

Let us consider $\vec{F}(x_1, x_2, ...x_n) = \vec{0}$ as the problem to be solved, for which $\vec{F}$ is a vector with m components. We are searching for the best vector (*i.e.*, the optimal vector, in a certain sense) $\vec{x} = (x_1, x_2, ...x_n)$ such that the equation is guaranteed. The basic idea of a *Newton-Raphson*-type method is to identify the zeros of the vector function $\vec{F}$, using the following algorithm :

Algorithm 11.1 *Newton-Raphson algorithm*

Initialize $\vec{x}$ randomly,
WHILE $\|F(\vec{x})\| \geq \varepsilon$
in first approximation, consider the asymptotic development of $\vec{F}$:

$$F_j(\vec{x} + d\vec{x}) = F_j(\vec{x}) + \frac{\partial F_j}{\partial x_1}dx_1 + \frac{\partial F_j}{\partial x_2}dx_2 + ... + \frac{\partial F_j}{\partial x_n}dx_n$$

where $j \in [1, m]$
solve the linear system $F_j(\vec{x} + d\vec{x}) = 0$ with $d\vec{x}$ as unknown
set $\vec{x}$ to the value $\vec{x} + d\vec{x}$
compute $F(\vec{x})$
END WHILE.
RETURN $\vec{x}$

Remark 11.17 *This algorithm yields a unique value for $\vec{x}$. This solution greatly depends on the value used to initialize $\vec{x}$. If more than one solution exists, only one is returned by the algorithm. Hence, the solution found may not be the global minimum of the function.*

Figure 11.11 illustrates the behavior of the algorithm for a function of one parameter. Solution x_3 is the unique solution returned by the algorithm, whereas three solutions can be shown.

Remark 11.18 *In some cases, the system to be solved is under-determined,* i.e., *$m \leq n$. In such cases, new equations need to be added, depending on the context, to find the best direction for the vector $\vec{F}$.*

11.4.3 Divide and conquer algorithm

Newton-Raphson-type algorithms do not allow a complete analysis in the interval in which the solution is assumed to be. The *divide and conquer* strategy attempts to identify the different sub-domains where a solution might exist. The main idea is to bound the values of $\vec{F}$ in an interval. If the interval contains 0, the latter is then subdivided into m (usually $m = 2$) sub-domains. This process is then repeated on each sub-domain. Each branch of the binary tree structure associated with this method is analyzed and recursively subdivided until a given minimal size is reached.

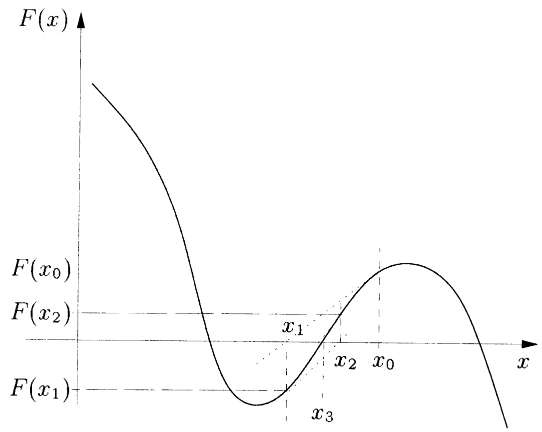

Figure 11.11: *Resolution of an equation using a Newton-Raphson method.*

Remark 11.19 *Such an algorithm does not strictly speaking solve the initial problem. In fact, it identifies the various sub-domains where a solution may be found. From a practical point of view, a Newton-Raphson algorithm is then employed to find the solution in each interval.*

Remark 11.20 (Method based on a grid) *This type of method starts by defining a grid* a priori *over the computational domain. Then, a linear interpolation of $\vec{F}$ is constructed. The non-linear problem is solved using the linear interpolation of $\vec{F}$. Seen from this point of view, this method is another way of subdividing the domain in such a way so as to localize the various possible solutions. From a practical point of view, the grid used is a regular grid in the space of $\vec{x}$.*