VIII. ALGEBRAIC GENERATION SYSTEMS

As noted earlier, the problem of generating a curvilinear coordinate system can be formulated as a problem of generating values of the cartesian coordinates in the interior of the rectangular transformed region from specified values on the boundaries. This, of course, can be done directly by interpolation from the boundaries, and such coordinate generation procedures are referred to as algebraic generation systems. Thus $\mathbf{r}(\xi^1,\xi^2,\xi^3)$ is given as a specific function of the curvilinear coordinates. This function contains certain coefficients which are determined so that the function matches specified values of the cartesian coordinates, and perhaps derivatives also, on the boundary and perhaps elsewhere. Evaluation of this interpolation function at constant values of the curvilinear coordinates then defines the coordinate system. Algebraic grid generation is discussed in Ref. [31] and [8], as well as in the surveys, Ref. [l], [5] and [37], and in detail in Ref. [32-36].

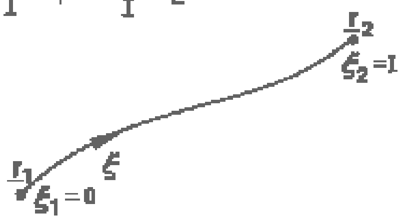
1. Unidirectional Interpolation

Unidirectional interpolation means the interpolation is in one curvilinear coordinate direction only. In this section the cartesian coordinate vector $\mathbf{r}$ will be shown as a function of the coordinate involved in the interpolation, as the unidirectional interpolation is fundamentally between points. These points can, however, lie on boundary (and perhaps interior) curves or surfaces, and in this sense the unidirectional interpolation can be considered to be between these curves or surfaces. Therefore the single-variable functional relationship $\mathbf{r}(\xi)$ used in this section can be considered to represent dependence on all coordinates, the interpolation points $\mathbf{r}_i$ being functions of the coordinates along the boundary curves or surfaces.

A. Lagrange interpolation

The simplest type of unidirectional interpolation is Lagrange interpolation, which is based on polynomials. In the linear form we have, with $0 \le \xi \le I$,

$$\mathbf{r}(\xi) = \left(1 - \frac{\xi}{I}\right)\mathbf{r}_1 + \frac{\xi}{I}\mathbf{r}_2 \tag{1}$$



Here $\mathbf{r}_1=\mathbf{r}(0)$ and $\mathbf{r}_2=\mathbf{r}(I)$, so that $\mathbf{r}(\xi)$ is defined in terms of the two boundary values, $\mathbf{r}_1$ and $\mathbf{r}_2$. The grid points are located at the successive integer values of ξ from 0 to I. One family of grid lines will be straight lines connecting corresponding boundary points with this linear interpolation.

The general form is

$$\underset{\sim}{r}(\xi) = \sum_{n=1}^{N} \phi_n\left(\frac{\xi}{I}\right) \underset{\sim}{r}_n \tag{2}$$

with $\underset{\sim}{r}_n = \underset{\sim}{r}(\xi_n)$, and the functions ϕ_n being polynomials defined on the entire interval $0 \leq \xi \leq I$ such that

$$\phi_n\left(\frac{\xi_m}{I}\right) = \delta_{nm} \tag{3}$$

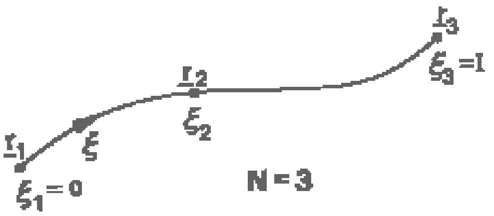
In the linear case given above we have, with N=2,

$$\phi_1\left(\frac{\xi}{I}\right) = 1 - \frac{\xi}{I} \quad \text{and} \quad \phi_2\left(\frac{\xi}{I}\right) = \frac{\xi}{I}$$

From Eq. (2) and (3),

$$\underset{\sim}{r}(\xi_m) = \sum_{n=1}^{N} \phi_n\left(\frac{\xi_m}{I}\right) \underset{\sim}{r}_n = \sum_{n=1}^{N} \delta_{nm} \underset{\sim}{r}_n = \underset{\sim}{r}_m$$

so that the interpolation function matches $\underset{\sim}{r}$ at the N points $\xi = \xi_1, \xi_2, ..., \xi_N = I$:



The specified interior points, $\underset{\sim}{r}_n$ for n=2,3,...N-1, are not necessarily grid points, since the grid points are defined by evaluating the interpolation formula at successive integer values of ξ, but are simply additional parameters that serve to control the distribution. It is possible to specify the locations of certain interior grids points, however, by taking the ξ_n corresponding to the specified $\underset{\sim}{r}_n$ to be the value of ξ at the grid point of interest.

The Lagrange interpolation polynomials, defined to satisfy by Eq. (3), are in general

$$\phi_n\left(\frac{\xi}{I}\right) = \prod_{l=1}^{N} \frac{\xi - \xi_l}{\xi_n - \xi_l} \qquad (l \neq n) \tag{4}$$

The quadratic forms thus are, with N=3, and $\xi_2 = I/2$.

$$\phi_1(\frac{\xi}{I}) = 2\,(\frac{\xi}{I} - \frac{1}{2})\,(\frac{\xi}{I} - 1)$$

$$\phi_2(\frac{\xi}{I}) = 4\,\frac{\xi}{I}\,(1 - \frac{\xi}{I})$$

$$\phi_3(\frac{\xi}{I}) = 2\,\frac{\xi}{I}\,(\frac{\xi}{I} - \frac{1}{2})$$

for which $\underset{\sim}{r}(\xi)$ is defined in terms of the two boundary values, $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_3$, and one interior value, $\underset{\sim}{r}_2$. It should be noted that the purpose of the inclusion of the interior points in grid generation is control of the grid point distribution, not to increase the accuracy of the interpolation as is normally the case. There is, in fact, no question of accuracy of the interpolation here, since the aim is just to generate a grid from the boundary values of the coordinates.

B. Hermite interpolation

Lagrange interpolation matches only function values. It is possible to match both function, $\underset{\sim}{r}$, and first-derivative, $\underset{\sim}{r}' = \underset{\sim}{r}_\xi$, values using Hermite interpolation defined by

$$\underset{\sim}{r}(\xi) = \sum_{n=1}^{N} \phi_n(\frac{\xi}{I})\,\underset{\sim}{r}_n + \sum_{n=1}^{N} \psi_n(\frac{\xi}{I})\,\underset{\sim}{r}_n' \qquad (5)$$

where the Hermite inerpolation polynomials are defined on $0 \le \xi \le I$ and satisfy the conditions

$$\phi_n(\frac{\xi_m}{I}) = \delta_{nm}, \quad \phi_n'(\frac{\xi_m}{I}) = 0$$

$$\psi_n(\frac{\xi_m}{I}) = 0, \quad \psi_n'(\frac{\xi_m}{I}) = \delta_{nm}$$

These polynomials can be obtained from the Lagrange interpolation polynomials by

$$\phi_n(\frac{\xi}{I}) = [1 - 2\phi_n'(\frac{\xi_n}{I})\,(\frac{\xi - \xi_n}{I})]\,\phi_n^2(\frac{\xi}{I}) \qquad (6a)$$

$$\psi_n(\frac{\xi}{I}) = (\frac{\xi - \xi_n}{I})\,\phi_n^2(\frac{\xi}{I}) \qquad (6b)$$

where the prime here indicates differentiation of the polynomial with respect to the argument, $\frac{\xi}{I}$. With N=2 we have

$$\Phi_1\left(\frac{\xi}{I}\right) = \left(1 + 2\frac{\xi}{I}\right)\left(1 - \frac{\xi}{I}\right)^2$$

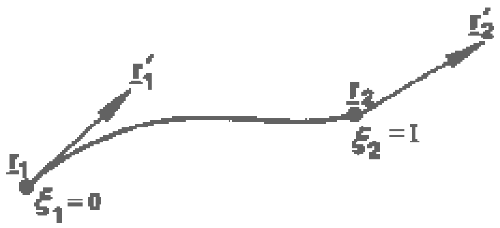
()

$$\Phi_2\left(\frac{\xi}{I}\right) = \left(3 - 2\frac{\xi}{I}\right)\left(\frac{\xi}{I}\right)^2$$

$$\Psi_1\left(\frac{\xi}{I}\right) = \left(1 - \frac{\xi}{I}\right)^2 \frac{\xi}{I}$$

$$\Psi_2\left(\frac{\xi}{I}\right) = \left(\frac{\xi}{I} - 1\right)\left(\frac{\xi}{I}\right)^2$$

and the function matches the two boundary values, $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$, and the first derivatives, $\underset{\sim}{r}'_1$ and $\underset{\sim}{r}'_2$, at the two boundaries.



Extensions of polynomial interpolation to match higher-order derivatives is obviously possible, the degree of the polynomial increasing with each additional condition or point to be matched. The polynomials of high degree exhibit considerable oscillation, however, so such procedures are not of great importance to grid generation. The general form again includes matches at interior points, which can be used to control the coordinate line spacing, since the first derivative, $\underset{\sim}{r}' = \underset{\sim}{r}_\xi$, is a measure of the grid point spacing here, with $\Delta\xi$ being unity between points by construction. As with Lagrange interpolation, these specified interior points may or may not be grid points.

It is also possible, of course, to omit points from either of the summations in Eq. (5), so that $\underset{\sim}{r}$ and its first derivatives are not both matched at all points (deficient Hermite interpolation). Thus, with N=2 and the n=1 term omitted from the second summation, the two boundary values would be matched, but the first dervative at only the ξ=I boundary would be matched. Clearly, the Hermite interpolation form, Eq. (5), could be equivalently defined in terms of the Lagrangian interpolation form, Eq. (2), with 2N points, since both are polynomial representations. Obviously either approach can be used to control the grid point spacing in the field.

The capability of specifying $\underset{\sim}{r}_\xi$, as well as $\underset{\sim}{r}$, can be used to make the grid orthogonal at the boundary. From Eq. (III-33) the unit normal to a ξ^i-coordinate surface is given by

$$\underset{\sim}{n}^i = \frac{\underset{\sim}{a}^i}{|\underset{\sim}{a}^i|} = \frac{\underset{\sim}{a}_j \times \underset{\sim}{a}_k}{|\underset{\sim}{a}_j \times \underset{\sim}{a}_k|} \qquad (i,j,k) \text{ cyclic}$$

Using Eq. (III-10) this becomes

$$\underset{\sim}{n}^i = \frac{\underset{\sim}{a}_j \times \underset{\sim}{a}_k}{\sqrt{g_{jj}g_{kk} - g_{jk}^2}}$$

The condition for orthogonally at the boundary then is that $\underset{\sim}{r}_{\xi^i}$ be in the direction of the unit normal to the boundary:

$$\underset{\sim}{r}_{\xi^i} = |\underset{\sim}{r}_{\xi^i}|\underset{\sim}{n}^i = \sqrt{g_{ii}}\,\frac{\underset{\sim}{a}_j \times \underset{\sim}{a}_k}{\sqrt{g_{jj}g_{kk} - g_{jk}^2}} \tag{7}$$

where $s^i = |\underset{\sim}{r}_{\xi^i}| = \sqrt{g_{ii}}$ is the spacing off the boundary to be specified. Since all the quantities with j and k subscripts can be evaluated from the points on the boundary, it remains only to specify the spacing, s^i, off the boundary and to use Eq. (7) for $\underset{\sim}{r}_{\xi^i}$ on the boundary in the Hermite expressions.

C. Other forms of polynomial interpolation

As noted above, Hermite interpolation, which matches $\underset{\sim}{r}$ and $\underset{\sim}{r}_\xi$ at N points, can be equivalently constructed as an interpolant which matches $\underset{\sim}{r}$ at 2N points. Another form of expression of the polynomial interpolation uses the direct expression of the polynomial, so that

$$\underset{\sim}{r}(\xi) = \sum_{n=0}^{N-1} \underset{\sim}{a}_n \left(\frac{\xi}{I}\right)^n \tag{8}$$

Here we must have

$$\underset{\sim}{a}_0 = \underset{\sim}{r}_0, \qquad \sum_{n=0}^{N-1} \underset{\sim}{a}_n = \underset{\sim}{r}_I$$

where $\underset{\sim}{r}_0$ and $\underset{\sim}{r}_I$ are the boundaries. This form is not as straightforward as the Lagrange form for use in grid generation, since in the latter form certain grid point locations can be specified directly, while in the former the coefficients must be evaluated in terms of these specified points.

Still another form is that of Bezier, using Bernstein polynomials:

$$\mathbf{r}(\xi) = \sum_{n=0}^{N} \frac{N!}{(N-n)!\,n!} \left(\frac{\xi}{I}\right)^{n} \left(1 - \frac{\xi}{I}\right)^{N-n} \mathbf{c}_n \qquad (9)$$

with

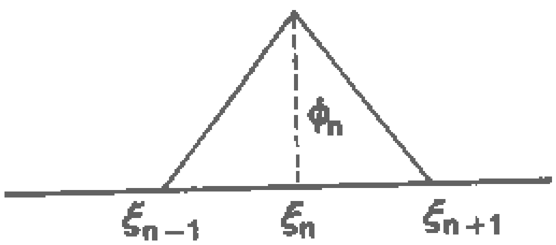
$$\mathbf{c}_0 = \mathbf{r}_0, \qquad \mathbf{c}_N = \mathbf{r}_I$$

Here we have

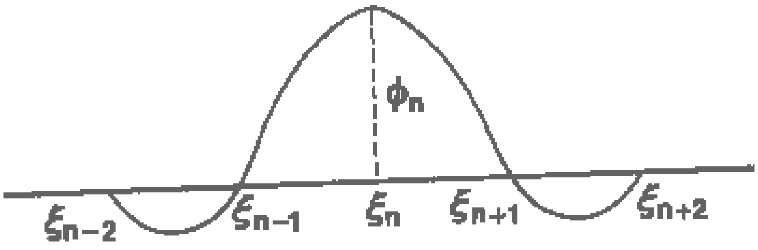
$$\mathbf{c}_1 - \mathbf{c}_0 = \frac{I}{N}\,\mathbf{r}_0', \qquad \mathbf{c}_N - \mathbf{c}_{N-1} = \frac{I}{N}\,\mathbf{r}_I'$$

Thus the coefficients $\mathbf{c}_1$ and $\mathbf{c}_{N-1}$ specify the slopes at the boundaries. An advantage of the Bezier form is that the coefficients define the vertices of an open polygon to which the curve is an approximation. Thus the general shape of the curve can be inferred by considering the coefficients to represent points in the field, with the lines from $\mathbf{c}_o$ to $\mathbf{c}_1$, and from $\mathbf{c}_{N-1}$ to $\mathbf{c}_N$, defining the slopes at the two ends. The shape of the curve can then be designed by the placement of the vertices in the field as indicated below. Modifications of the curve can thus be made by adjusting the positions of these vertices.

Still another form can be defined using piecewise polynomials for the interpolation functions. Some degree of continuity must be lost in this case, of course. Continuity of the grid lines can be achieved using the piecewise-linear polynomials shown below (truncated versions apply at or near the end points):



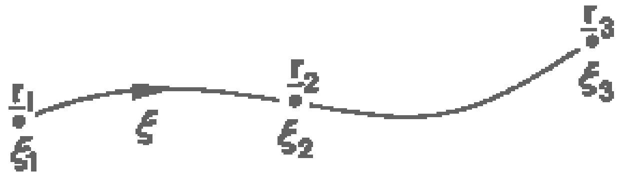
while slope continuity can be gotten with the following piecewise polynomials:



Such piecewise polynomials allow a greater degree of local adjustment to be made, since the polynomial ϕ_n which multiplies the interpolation point $\underset{\sim}{r}_n$ vanishes except in the immediate vicinity of $\underset{\sim}{r}_n$. By the conditions (3), any interpolation function ϕ_n must vanish at all the interpolation points except $\underset{\sim}{r}_n$, but need not vanish between the points. Adding more interpolation points with global polynomials thus means increasing the degree of the polynomials, since the numbers of zeros must increase, and hence the polynomial becomes highly oscillitory.

D. Splines

The Lagrange and Hermite interpolation functions given above are completely continuous at all points. Complete continuity, however, may be attained at the price of oscillation. Both of these forms fit a single polynomial from one boundary to the other, matching specified values of the coordinates and perhaps the derivatives thereof (i.e., the point spacing). As more interior points are included, or as the first derivatives are included, the order of this global polynomial increases and thus oscillations become more likely. An alternative approach is to fit a low-order polynomial between each of the specified interior points, with continuity of as many derivatives as is possible enforced at the interior points. The interpolation function is then a piecewise-continuous polynomial.



This type of interpolation function is called a spline and is formed as follows for the most common case of the cubic spline.

With a cubic polynomial fitted between points $\underset{\sim}{r}_i$ and $\underset{\sim}{r}_{i+1}$ we have a linear variation of the second derivative between these points and thus

$$\underset{\sim}{r}''(\xi) = \frac{\xi_{i+1} - \xi}{\xi_{i+1} - \xi_i}\underset{\sim}{r}_i'' + \frac{\xi - \xi_i}{\xi_{i+1} - \xi_i}r_{i+1}'' \qquad \xi_i \leq \xi \leq \xi_{i+1} \tag{10}$$

After two integrations and evaluation of the two constants ofintegration such that $\underset{\sim}{r}(\xi_i)=\underset{\sim}{r}_i$ and $\underset{\sim}{r}(\xi_{i+1})=\underset{\sim}{r}_{i+1}$, we have on $\xi_i \le \xi \le \xi_{i+1}$

$$\underset{\sim}{r}(\xi) = \frac{(\xi_{i+1} - \xi)^3}{6(\xi_{i+1} - \xi_i)}\underset{\sim}{r}_i'' + \frac{(\xi - \xi_i)^3}{6(\xi_{i+1} - \xi_i)}\underset{\sim}{r}_{i+1}''$$
$$+ \left(\frac{1}{\xi_{i+1} - \xi_i}\underset{\sim}{r}_i - \frac{\xi_{i+1} - \xi_i}{6}\underset{\sim}{r}_i''\right)(\xi_{i+1} - \xi) \tag{11}$$
$$+ \left(\frac{1}{\xi_{i+1} - \xi_i}\underset{\sim}{r}_{i+1} - \frac{\xi_{i+1} - \xi_i}{6}\underset{\sim}{r}_{i+1}''\right)(\xi - \xi_i)$$

Then, after differentiation and setting $\xi = \xi_i$, we have on $\xi_i \le \xi \le \xi_{i+1}$,

$$\underset{\sim}{r}_i' = -\frac{\xi_{i+1} - \xi_i}{6}(2\underset{\sim}{r}_i'' + \underset{\sim}{r}_{i+1}'') + \frac{1}{\xi_{i+1} - \xi_i}(\underset{\sim}{r}_{i+1} - \underset{\sim}{r}_i) \tag{12}$$

Similar evaluation on the adjacent interval $\xi_{i-1} \le \xi \le \xi_i$ gives

$$\underset{\sim}{r}_i' = \frac{\xi_i - \xi_{i-1}}{6}(2\underset{\sim}{r}_i'' + \underset{\sim}{r}_{i-1}'') + \frac{1}{\xi_i - \xi_{i-1}}(\underset{\sim}{r}_i - \underset{\sim}{r}_{i-1}) \tag{13}$$

Equating these two expressions in order to produce continuity of r’ at the interior points, we have

$$(\xi_i - \xi_{i-1})\underset{\sim}{r}_{i-1}'' + 2(\xi_{i+1} - \xi_{i-1})\underset{\sim}{r}_i'' + (\xi_{i+1} - \xi_i)\underset{\sim}{r}_{i+1}''$$
$$= 6\left(\frac{\underset{\sim}{r}_{i+1} - \underset{\sim}{r}_i}{\xi_{i+1} - \xi_i} - \frac{\underset{\sim}{r}_i - \underset{\sim}{r}_{i-1}}{\xi_i - \xi_{i-1}}\right) \tag{14}$$

which is a tridiagonal equation for $\underset{\sim}{r}''$ at the interior points. It is necessary to set some conditions on $\underset{\sim}{r}''$ on the boundaries in order to solve this system, and the "natural" spline uses $\underset{\sim}{r}_1'' = \underset{\sim}{r}_I'' = 0$. This choice minimizes the total curvature, and thus the natural spline is the smoothest interpolant. This solution defines the $\underset{\sim}{r}_i$ in terms of the $\underset{\sim}{r}_i$, so that substitution of these values for $\underset{\sim}{r}_i''$ into Eq. (11) then gives the spline in the general form of Eq. (2), except that the interpolation functions, ϕ_n, are, of course, different from the Lagrange interpolation polynomials. It should be recalled again that the interior points may or may not be grid points, the latter being defined by the interpolation formula evaluated at successive integer values of ξ after the spline has been constructed over the entire field.

E. Tension splines

The spline tends to give a very smooth point distribution. Stronger localized curvature around the specified interior points can be obtained with the tension spline. Here Eq. (10) on $\xi_i \le \xi \le \xi_{i+1}$ is replaced by

$$\mathbf{r}''(\xi) - \sigma^2 \mathbf{r}(\xi) = \frac{\xi_{i+1} - \xi}{\xi_{i+1} - \xi_i}(\mathbf{r}_i'' - \sigma^2 \mathbf{r}_i) + \frac{\xi - \xi_i}{\xi_{i+1} - \xi_i}(\mathbf{r}_{i+1}'' - \sigma^2 \mathbf{r}_{i+1}) \qquad (15)$$

where σ^2 is a constant to be specified. (The tension spline tends progressively toward a linear function for large values of σ, and toward a cubic spline for small values.) Integration and evaluation of constants then yields, on $\xi_i \le \xi \le \xi_{i+1}$,

$$\mathbf{r}(\xi) = \frac{1}{\sigma^2}\frac{\sinh[\sigma(\xi_{i+1} - \xi)]}{\sinh[\sigma(\xi_{i+1} - \xi_i)]}\mathbf{r}_i'' + \frac{1}{\sigma^2}\frac{\sinh[\sigma(\xi - \xi_i)]}{\sinh[\sigma(\xi_{i+1} - \xi_i)]}\mathbf{r}_{i+1}'' + \left(\mathbf{r}_i - \frac{1}{\sigma^2}\mathbf{r}_i''\right)\frac{\xi_{i+1} - \xi}{\xi_{i+1} - \xi_i} + \left(\mathbf{r}_{i+1} - \frac{1}{\sigma^2}\mathbf{r}_{i+1}''\right)\frac{\xi - \xi_i}{\xi_{i+1} - \xi_i} \qquad (16)$$

The requirement of continuity of first derivatives at the interior points then yields the tridiagonal equation

$$\left[\frac{1}{\Delta_{i-1}} - \frac{\sigma}{\sinh(\sigma\Delta_{i-1})}\right]\mathbf{r}_{i-1}'' + \left[\frac{1}{\Delta_i} - \frac{\sigma}{\sinh(\sigma\Delta_i)}\right]\mathbf{r}_{i+1}'' + \left[\frac{\sigma}{\tanh(\sigma\Delta_{i-1})} - \frac{1}{\Delta_{i-1}} + \frac{\sigma}{\tanh(\sigma\Delta_i)} - \frac{1}{\Delta_i}\right]\mathbf{r}_i'' = \sigma^2\left(\frac{\mathbf{r}_{i+1} - \mathbf{r}_i}{\Delta_i} - \frac{\mathbf{r}_i - \mathbf{r}_{i-1}}{\Delta_{i-1}}\right) \qquad (17)$$

where $\Delta_i = \xi_{i+1} - \xi_i$ and $\Delta_{i-1} = \xi_i - \xi_{i-1}$. Some application of tension splines are given in Ref. [33].

F. B-Splines

One further possibility is to use piecewise continuous functions which satisfy the cardinality conditions by vanishing identically outside some interval around ξ_n, as discussed in Section C above. This type of function allows the interpolation to be modified locally without affecting the interpolation function elsewhere. The B-splines are an example of this approach.

From Eq. (14) a cubic spline which matches the function at N points, with continuity of second-derivatives, requires N+2 items of data, i.e., the N values of $\underline{r}_n$ (n=1,2,...N) and the values of $\underline{r}''$ at each boundary. Therefore a cubic spline which has $\underline{r} = \underline{r}' = \underline{r}'' = 0$ at each boundary can be defined over five points if $\underline{r}$ is specified at only a single interior point (since N+2=7 data items can be specified here). If such a spline over five points is joined to the line $\underline{r} = 0$ outside these five points, we have a function which is non-zero only over four intervals and yet which has continuous second derivatives everywhere. Such a function is called a B-spline, denoted $N_{4N}(\underline{\xi})$, where the end-points of the non-zero interval are ξ_{n-4} and ξ_n. Similarly, quadratic, linear and constant B-splines are non-zero over three, two, and one intervals, respectively, and are denoted N_{qn}, where q = 3, 2, and 1. The end-points of the interval of non-zero values for these splines are ξ_{n-q} and ξ_n. The specification of a single value in this interval is usually replaced by the specification of the integral over the interval so that

$$\int_{\xi_{n-q}}^{\xi_n} N_{qn}(\xi)d\xi = \frac{\xi_n - \xi_{n-q}}{q} \tag{18}$$

The practical importance of B-splines is that any spline of order q (the cubic spline is of order 4) can be expressed as a sum of multiples of B-splines. Thus the cubic spline can be written as

$$\underline{r}(\xi) = \sum_{n=0}^{N} \underline{c}_n N_{4n}(\xi) \tag{19}$$

Since the B-splines are non-zero only over four intervals, the modification of one coefficient here only affects the function over four intervals, thus allowing more localized control of the resulting grid.

The B-splines can be calculated from the recurrence relation

$$N_{qn}(\xi) = \frac{(\xi - \xi_{n-q}) N_{q-1,n-1}(\xi) + (\xi_n - \xi) N_{q-1,n}(\xi)}{\xi_{n-1} - \xi_{n-q}} \tag{20}$$

Thus $N_{4n}(\xi)$ requires the successive calculation of $N_{1,n-1}$, $N_{2,n-1}$, $N_{2,n}$, $N_{3,n-1}$, $N_{3,n}$, and finally $N_{4,n}$. The constant B-spline, $NN_{1,n-1}$, used to start this calculation, is given on the interval $\xi_{n-2} \le \xi \le \xi_{n-1}$ by $N_{1,n-1}=1$ and vanishes elsewhere.

For the point $\underset{\sim}{r}_n$ we have, in view of the vanishing of the B-splines outside four intervals,

$$\underset{\sim}{r}_n = \underset{\sim}{c}_{n-1} N_{4,n-1}(\xi_n) + \underset{\sim}{c}_n N_{4,n}(\xi_n) + \underset{\sim}{c}_{n+1} N_{4,n+1}(\xi_n) \tag{21}$$

which is a tridiagonal relation (N+1 equations) for the coefficients $\underset{\sim}{c}_n$, $\underset{\sim}{c}_o=\underset{\sim}{r}_o$ and $\underset{\sim}{c}_N=\underset{\sim}{r}_N$. Thus, even though the modification of a single coefficient only affects four intervals, the modification of an interpolation point requires a re-determination of all the coefficients and thus affects the function over the entire range.

The coefficients, $\underset{\sim}{c}_n$, in the B-spline representation may be interpreted as the vertices of an open polygon, to which the curve is an approximation, as for the Bezier form discussed above. The slopes at the ends are defined by the directions $\underset{\sim}{c}_1-\underset{\sim}{c}_o$ and $\underset{\sim}{c}_N-\underset{\sim}{c}_{N-1}$. The curve passes close to the mid-point of each side, with the exception of the first and last sides. The curve also passes through the points $(\underset{\sim}{r}_{k-1} + 4\underset{\sim}{r}_k + \underset{\sim}{r}_{k+1})/6$ for k = 2,3,...,N-2. These points are one-third of the way along the straight line joining $\underset{\sim}{r}_k$ to the mid-point of the line joining $\underset{\sim}{r}_{k-1}$ and $\underset{\sim}{r}_{k+1}$. Since the B-splines are non-zero only on four intervals, the alteration of one vertex only affects the curve in its immediate vicinity. An application of B-splines in grid generation is given in Ref. [39].

G. Multi-surface interpolation

The multi-surface method, discussed in Ref. [32]-[36], is also a unidirectional interpolation procedure. This procedure is constructed from an interpolation of a specified vector field, followed by vector normalizations at each interpolation point in order to cause a desired telescopic collapse so that the boundaries are matched. The specified vector field is defined from piecewise-linear curves determined by the boundaries and successive intermediate control surfaces. Normals to such surfaces are special cases. Polynomial interpolants for the vector field yield all of the classical polynomial cases along with a rational method for avoiding disasters such as can occur with direct Hermite interpolation with excessively large or discontinuous derivatives. Here the immediate surfaces are not coordinate surfaces, but are used only to define the vector field. These vectors are taken to be tangents to the coordinate lines intersecting the surfaces, so that integration of this vector field produces the position vector field for the grid points.

A collection of subroutines which automatically perform the necessary parts of grid construction using this multi-surface procedure has been written and is described in Ref. [34]. Some of the automation features of this collection are applicable to other grid construction procedures as well. These subroutines can rotate and move curves, prospect one curve from another, normalize and parameterize curves, cluster points on a curve, and perform other such utilitarian functions to aid in the setup of an overall configuration.

In the multi-surface interpolation we have

$$\mathbf{r}(\xi) = \mathbf{r}_0 + \sum_{n=1}^{N-1} \frac{G_n(\xi)}{G_n(I)} (\mathbf{P}_{n+1} - \mathbf{P}_n) \tag{22a}$$

where

$$G_n(\xi) = \int_0^{\xi} \psi_n\left(\frac{\xi'}{I}\right) d\xi' \tag{22b}$$

and where the $\underset{\sim}{P}_n$ are specified points, with $\underset{\sim}{P}_1=\underset{\sim}{r}(0)$ and $\underset{\sim}{P}_N=\underset{\sim}{r}(I)$, on the boundary surfaces. (Recall the discussion at the beginning of this section, i.e., that the points can be considered to lie on curves or surfaces and thus the interpolation, while being fundamentally between points, can be considered to be between the surfaces on which those points lie.) Here the telescopic collapse for the series for ξ =I matches the boundary at $\underset{\sim}{r}_I$. The intermediate points here, $\underset{\sim}{P}_2, \underset{\sim}{P}_3, \ldots, \underset{\sim}{P}_{N-1}$, are not grid points, but serve only to define the slopes $\underset{\sim}{r}_\xi$, as given by Eq. (24) below. Eq. (22) is a polynomial if the functions ψ_n are polynomials, but such is not required.

Since, by differentiation of Eq. (22),

$$\mathbf{r}_\xi = \sum_{n=1}^{N-1} \frac{\psi_n\left(\frac{\xi}{I}\right)}{G_n(I)} (\underset{\sim}{P}_{n+1} - \underset{\sim}{P}_n) \tag{23}$$

we have, for $0=\xi_1<\xi_2<\ldots<\xi_{n-1}=I$,

$$\underset{\sim}{r}_\xi(\xi_n) = \frac{\underset{\sim}{P}_{n+1} - \underset{\sim}{P}_n}{G_n(I)} \tag{24}$$

if the functions ψ_n satisfy the cardinality conditions

$$\psi_n\left(\frac{\xi_m}{I}\right) = \delta_{nm} \qquad n = 1,2,\ldots,N-1, \qquad m = 1,2,\ldots,N-1 \tag{25}$$

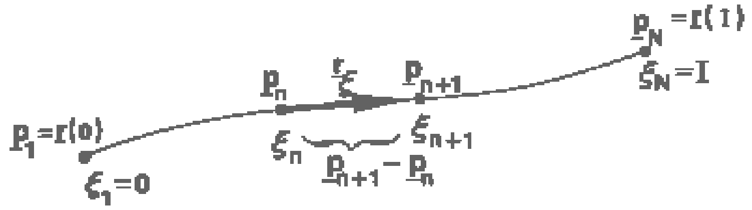
The polynomials that satisfy these conditions are simply the Lagrange polynomials given by Eq. (4), here stated as

$$\psi_n\left(\frac{\xi}{I}\right) = \prod_{l=1}^{N-1} \frac{\xi - \xi_l}{\xi_n - \xi_l} \qquad (l \neq n) \tag{26}$$

Using Eq. (24), Eq. (23) can be written as

$$\underset{\sim}{r}_{\xi}(\xi) = \sum_{n=1}^{N-1} \psi_n\left(\frac{\xi}{I}\right) \underset{\sim}{r}_{\xi}(\xi_n) \tag{27}$$

This form thus is based on an interpolation of the first-derivatives $\underset{\sim}{r}_{\xi}$, instead of $\underset{\sim}{r}$, the interpolation expression for $\underset{\sim}{r}$ coming from an integration of the interpolation function for $\underset{\sim}{r}_{\xi}$. Note, however, that this amounts to the specification of the slope $\underset{\sim}{r}_{\xi}$ at particular values of the curvilinear coordinate ξ_n, and not at a specified position in space as is done in the Hermite form. It is clear from Eq. (24) that the intermediate points, $\underset{\sim}{P}_n$ for n=2,3,....,N-2, serve to define the slopes $\underset{\sim}{r}_{\xi}(\xi_n)$:



Because of the integration involved, the degree of the interpolation polynomial will be one greater than that of the functions ψ_n.

Also with Eq. (24), the interpolation for $\underset{\sim}{r}$, Eq. (22), can be written

$$\underset{\sim}{r}(\xi) = \underset{\sim}{r}(0) + \sum_{n=1}^{N-1} G_n(\xi) \underset{\sim}{r}_{\xi}(\xi_n) \tag{28}$$

and thus is equivalent to a form of deficient Hermite interpolation. In implementation, however, it is the points $\underset{\sim}{P}_n$ that are specified, as in Eq. (22). Again it should be recalled that the point $\underset{\sim}{P}_n$ is not the grid point at ξ_n, exécept for the boundaries ξ_1=0 and ξ_N=I.

As has been noted, $\underset{\sim}{P}_1$ and $\underset{\sim}{P}_n$ are determined by the boundaries:

$$\underset{\sim}{P}_1 = \underset{\sim}{r}(0), \qquad \underset{\sim}{P}_N = \underset{\sim}{r}(I) \tag{29a}$$

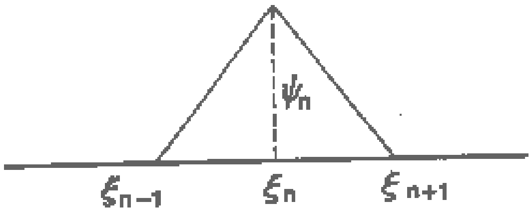
For N$\geq$4, $\underset{\sim}{P}_2$ and $\underset{\sim}{P}_{N-1}$ are determined by the intended values of $\underset{\sim}{r}_{\xi}$ at the boundaries through Eq. (24):

$$\begin{aligned} \underset{\sim}{P}_2 &= \underset{\sim}{r}(0) + \underset{\sim}{r}_{\xi}(0) G_1(I) \\ \underset{\sim}{P}_{N-1} &= \underset{\sim}{r}(I) - \underset{\sim}{r}_{\xi}(I) G_{N-1}(I) \end{aligned} \tag{29b}$$

For N=3 only one of the above equations can be used, i.e., $\underset{\sim}{r}_{\xi}$ can be specified at either

For N=3 only one of the above equations can be used, i.e., ξ can be specified at either boundary but not at both. The use of the intermediate surfaces, instead of direct specification of the derivatives $\underset{\sim}{r}_\xi$ as in classical Hermite interpolation, provides a geometric interpretation that serves to help avoid the overlapping of grid lines that can occur if too large a value is given for $\underset{\sim}{r}_\xi$.

Following Ref. [35], consider now for the ψ_n the piecewise-linear functions diagramed below:



with the normalization

$$\psi_1(\xi_1) = \frac{2}{\Delta\xi_1}, \qquad \psi_{N-1}(\xi_{N-1}) = \frac{2}{\Delta\xi_{N-2}}$$

$$\psi_n(\xi_n) = \frac{2}{\Delta\xi_{n-1} + \Delta\xi_n} \qquad n = 2,3,\ldots\ldots,N-2$$

where $\Delta\xi_n = \xi_{n+1} - \xi_n$ so that each ψ_n integrates to unity. These functions are given by, for n=2,3,....,N-2,

$$
\psi_1(\xi) = \begin{cases} \dfrac{2}{\Delta\xi_1^2}(\xi_2 - \xi) & \xi_1 \le \xi \le \xi_2 \\[2ex] 0 & \xi_2 \le \xi \le \xi_{N-1} \end{cases}
$$

$$
\psi_{N-1}(\xi) = \begin{cases} 0 & \xi_1 \le \xi \le \xi_{N-2} \\[2ex] \dfrac{2}{\Delta\xi_{N-2}^2}(\xi - \xi_{N-2}) & \xi_{N-2} \le \xi \le \xi_{N-1} \end{cases} \tag{30}
$$

$$
\psi_n(\xi) = \begin{cases} 0 & \xi_1 \le \xi \le \xi_{n-1} \\[1ex] \dfrac{2}{\Delta\xi_{n-1} + \Delta\xi_n}\left(\dfrac{\xi - \xi_n}{\Delta\xi_{n-1}} + 1\right) & \xi_{n-1} \le \xi \le \xi_n \\[2ex] \dfrac{2}{\Delta\xi_{n-1} + \Delta\xi_n}\left(\dfrac{\xi_n - \xi}{\Delta\xi_n} + 1\right) & \xi_n \le \xi \le \xi_{n+1} \\[1ex] 0 & \xi_{n+1} \le \xi \le \xi_{N-1} \end{cases}
$$

With these functions we have

$$G_1(\xi) = \begin{cases} 1 - \dfrac{(\xi_2 - \xi)^2}{\Delta\xi_1^2} & \xi_1 \le \xi \le \xi_2 \\ 1 & \xi_2 \le \xi \le \xi_{N-1} \end{cases}$$

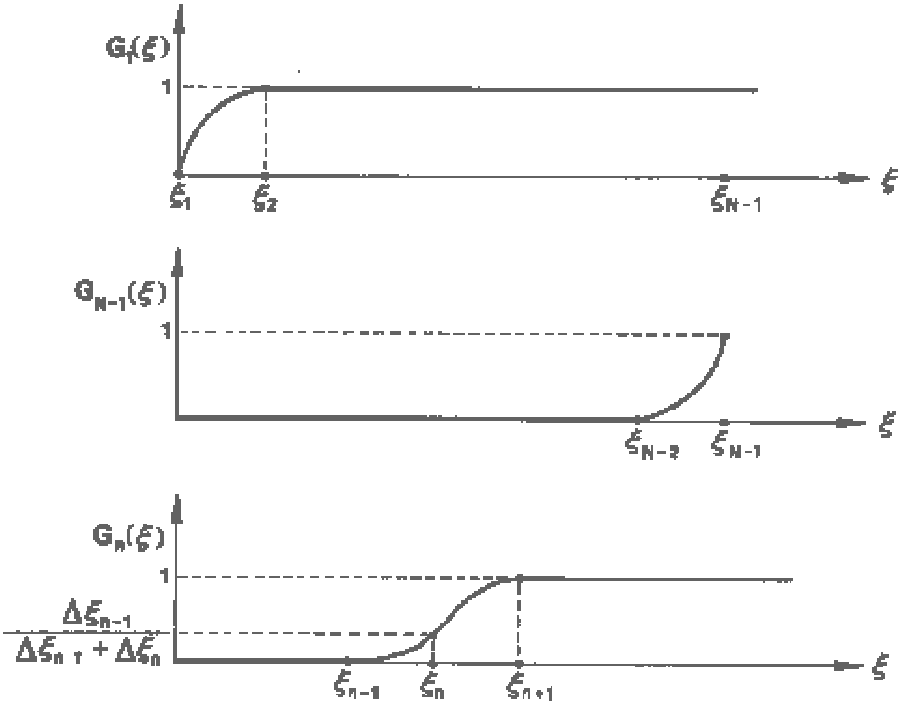
$$G_{N-1}(\xi) = \begin{cases} 0 & \xi_1 \le \xi \le \xi_{N-2} \\ \dfrac{(\xi - \xi_{N-2})^2}{\Delta\xi_{N-2}^2} & \xi_{N-2} \le \xi \le \xi_{N-1} \end{cases} \tag{31}$$

$$G_n(\xi) = \begin{cases} 0 & \xi_1 \le \xi \le \xi_{n-1} \\ \dfrac{(\xi - \xi_{n-1})^2}{(\Delta\xi_{n-1} + \Delta\xi_n)\Delta\xi_{n-1}} & \xi_{n-1} \le \xi \le \xi_n \\ 1 - \dfrac{(\xi_{n+1} - \xi)^2}{(\Delta\xi_{n-1} + \Delta\xi_n)\Delta\xi_n} & \xi_n \le \xi \le \xi_{n+1} \\ 1 & \xi_{n+1} \le \xi \le \xi_{N-1} \end{cases}$$

Note that $G_n(I) = 1$ here for all n and that $G_n(\xi_1) = 0$,

$$G_{N-1}(\xi_{N-1}) = 1 \text{ and } G_n(\xi_n) = \frac{\Delta\xi_{n-1}}{\Delta\xi_{n-1} + \Delta\xi_n}$$

These interpolation functions have the form



Now on the interval $\xi_n \leq \xi \leq \xi_{n+1}$ we have

$G_m(\xi)=1$ for m=1,2,....,n-1 and $G_m(\xi)=0$ for m=n+2,n+3,....,N-1. Therefore, on this interval

$$r(\xi) = r(0) + \sum_{i=1}^{n-1} (r_{i+1} - r_i) + G_n(\xi)\,(r_{n+1} - r_n)$$

$$+ G_{n+1}(\xi)\,(r_{n+2} - r_{n+1})$$

which, because of the telescopic collapse of the summation, reduces to, for $\xi_n \leq \xi \leq \xi_{n+1}$,

$$r(\xi) = r_n + G_n(\xi)\,(r_{n+1} - r_n) + G_{n+1}(\xi)\,(r_{n+2} - r_{n+1}) \tag{32}$$

Then

$$r(\xi_n) = \frac{(\Delta\xi_n)r_n + (\Delta\xi_{n-1})r_{n+1}}{\Delta\xi_{n-1} + \Delta\xi_n} \tag{33a}$$

and

$$r(\xi_{n+1}) = \frac{(\Delta\xi_{n+1})r_{n+1} + (\Delta\xi_n)r_{n+2}}{\Delta\xi_n + \Delta\xi_{n+1}} \tag{33b}$$

Also, from Eq. (32), for $\xi_n \leq \xi \leq \xi_{n+1}$

$$\underset{\sim}{r}_\xi(\xi) = \frac{2(\xi_{n+1} - \xi)}{(\Delta\xi_{n-1} + \Delta\xi_n)\Delta\xi_n}(\underset{\sim}{P}_{n+1} - \underset{\sim}{P}_n) + \frac{2(\xi - \xi_n)}{(\Delta\xi_n + \Delta\xi_{n+1})\Delta\xi_n}(\underset{\sim}{P}_{n+2} - \underset{\sim}{P}_{n+1}) \tag{34}$$

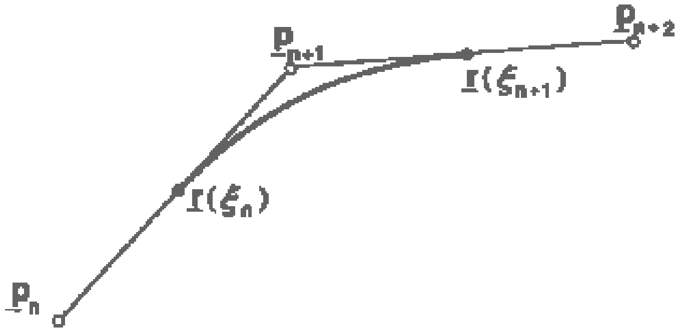
so that

$$\underset{\sim}{r}_\xi(\xi_n) = \frac{2}{\Delta\xi_{n-1} + \Delta\xi_n}(\underset{\sim}{P}_{n+1} - \underset{\sim}{P}_n) \tag{35a}$$

and

$$\underset{\sim}{r}_\xi(\xi_{n+1}) = \frac{2}{\Delta\xi_n + \Delta\xi_{n+1}}(\underset{\sim}{P}_{n+2} - \underset{\sim}{P}_{n+1}) \tag{35b}$$

We thus have on this interval



Thus on the interval $\xi_n \leq \xi \leq \xi_{n+1}$, $\underset{\sim}{r}(\xi)$ is affected only by $\underset{\sim}{P}_n$, $\underset{\sim}{P}_{n+1}$, and $\underset{\sim}{P}_{n+2}$. Conversely $\underset{\sim}{P}_n$ affects only the grid point locations on the interval $\xi_{n-2} \leq \xi \leq \xi_{n+1}$. Therefore local adjustments in the grid point locations can be made without affecting all of the points.

With the grid points located at unity increments of ξ, so that I is the total number of points, we have, from Eq. (33), the grid points given by

$$\underset{\sim}{r}_n = \frac{1}{2}(\underset{\sim}{P}_n + \underset{\sim}{P}_{n+1})$$

The local control provided by these piecewise linear interpolants can be used to restrict undesirable mesh forms or to embed desirable ones within a global system with continuous first derivatives (cf. Ref. [34]-[35]). The second derivatives are, however, discontinuous. As examples, the propagation of boundary slope discontinuities can be arbitrarily restricted and general rectilinear Cartesian systems can be embedded to simplify problems over a large part of their domain.

In a further development (cf. Ref. [36]) the procedure is extended to use piecewise quadratic local interpolants, thus achieving continuity of second derivatives, with discontinuous third derivatives. The conceptual extension to higher order piecewise polynomial local interpolants, with consequent higher degree of continuity, is also discussed. Note that because of the integration of ψ_n, the level of derivative continuity is always one greater than that of the piecewise polynomials.

H. Uniformity

It may be desirable for purposes of control of the grid point distribution to have a uniform distribution of the relative pro)ection of $\mathbf{r}(\xi)-\mathbf{r}(0)$ along the straight line connecting the boundary points, i.e., $\mathbf{r}(I)-\mathbf{r}(0)$. This property has been called "uniformity" by Eiseman, cf. Ref. [32] - [36], and can be realized as follows: The unit vector along this straight line is

$$\frac{\mathbf{r}(I) - \mathbf{r}(0)}{|\mathbf{r}(I) - \mathbf{r}(0)|}$$

so that the relative pro)ection of $\mathbf{r}(\xi)-\mathbf{r}(0)$ along this line is given by

$$S(\xi) = \frac{[\mathbf{r}(\xi) - \mathbf{r}(0)] \cdot \dfrac{\mathbf{r}(I) - \mathbf{r}(0)}{|\mathbf{r}(I) - \mathbf{r}(0)|}}{|\mathbf{r}(I) - \mathbf{r}(0)|} = [\mathbf{r}(\xi) - \mathbf{r}(0)] \cdot \mathbf{I} \tag{36}$$

where

$$\mathbf{I} = \frac{\mathbf{r}(I) - \mathbf{r}(0)}{|\mathbf{r}(I) - \mathbf{r}(0)|^2} \tag{37}$$

Uniformity is then achieved by choosing the interpolation parameters such that $S(\xi)$ is linear. This does not completely determine all the interpolation parameters, however, so that some remain to be specified as desired. Uniformity is trivially assumed for linear interpolation of course.

For Lagrange interpolation we have, from Eq. (2),

$$S(\xi) = \sum_{n=2}^{N} \phi_n\left(\frac{\xi}{I}\right) [\mathbf{r}_n - \mathbf{r}(0)] \cdot \mathbf{I} \tag{38}$$

so that uniformity is achieved by selecting the $\mathbf{r}_n$, for n=2,3,....,N-1, to cause all of the terms in S that are quadratic or higher to vanish. For Hermite interpolation, (5),

$$S(\xi) = \sum_{n=2}^{N} \phi_n\left(\frac{\xi}{I}\right) [\mathbf{r}_n - \mathbf{r}(0)] \cdot \mathbf{I} + \sum_{n=1}^{N} \psi_n\left(\frac{\xi}{I}\right) \mathbf{r}_n' \cdot \mathbf{I} \tag{39}$$

For the multi-surface interpolation defined by Eq. (22) we have

$$S(\xi) = \sum_{n=1}^{N-1} \frac{G_n(\xi)}{G_n(I)} (\underline{P}_{n+1} - \underline{P}_n) \cdot I \tag{40}$$

For $S(\xi)$ to be linear we must have

$$S(\xi) = \xi$$

or

$$\sum_{n=1}^{N-1} \frac{\psi_n(\frac{\xi}{I})}{G_n(I)} (\underline{P}_{n+1} - \underline{P}_n) \cdot I = \text{constant} \tag{41}$$

But, using Eq. (25), we then have

$$(\underline{P}_{n+1} - \underline{P}_n) \cdot I = \frac{G_n(I)}{\psi_n(\frac{\xi_n}{I})} \tag{42}$$

as the uniformity condition on the $\underset{\sim}{P}_n$'s (cf. Ref. [35]). Both the $\underset{\sim}{P}_n$ (for n=2,3,....,N-1) and the ξ_n (for n=2,3,....,N-2) are free to be chosen in order to satisfy the uniformity conditions (42). Thus a one-parameter family of cubic forms (N=4) results, a two-parametric family of quartic forms, etc. Substitution of Eq. (42) back into (41) yields a restriction on the choice of the functions ψ_n since these must satisfy the relation

$$\sum_{n=1}^{N-1} \frac{\psi_n(\frac{\xi}{I})}{\psi_n(\frac{\xi_n}{I})} = 1 \tag{43}$$

Uniformity is particularly useful when the distribution function, such as those discussed in the next section, is used to redistribute the points on the grid lines set up by the interpolation (cf. Ref. [34]-[36]). Thus the interpolation is first applied with I=1 and with the uniformity conditions enforced. The final grid points then are placed according to the distribution function on the grid lines set up by the interpolation variable in place of the arc length, s, in the distribution function $s(\xi)$.

I. Functions other than polynomials

The interpolation functions in the general forms given by Eq. (2), (5), and (22) do not have to be polynomials, and, in fact, if the variation in spacing over the field is large, other functions are better suited for grid generation. With N=2, Eq. (2) can be written in the form

$$\underline{r}(\xi) = \phi(\tfrac{\xi}{I})\, \underline{r}_2 + [1 - \phi(\tfrac{\xi}{I})]\, \underline{r}_1 \tag{44}$$

where ϕ can be any function such that $\phi(0)=0$ and $\phi(1)=1$. Here we have taken $\phi_1=1-\phi$ and

$\phi_2 = \phi$. The linear polynomial case is obtained here with $\phi(\frac{\xi}{I}) = \frac{\xi}{I}$. The function ϕ in this form may contain parameters which can be determined so as to match the slope at the boundary, or to match interior points and slopes.

The interpolation function, ϕ, in this form is often referred to as a "stretching" function, and the most widely used function has been the exponential:

$$\phi\left(\frac{\xi}{I}\right) = \frac{\exp\left(\frac{\alpha\xi}{I}\right) - 1}{\exp(\alpha) - 1} \tag{45}$$

where α is a parameter that can be determined to match the slope at a boundary. Thus, since, from Eq. (44)

$$\mathbf{r}_\xi = \frac{1}{I}(\mathbf{r}_2 - \mathbf{r}_1)\,\phi' \tag{46}$$

we can determine α from the equation

$$(\mathbf{r}_\xi)_1 = \frac{\mathbf{r}_2 - \mathbf{r}_1}{I}\,\frac{\alpha}{\exp(\alpha) - 1} \tag{47}$$

with $(\mathbf{r}_\xi)_1$ specified.

As noted in Chapter V, the truncation error is strongly affected by the point distribution, and studies of distribution functions have been made in that regard. The exponential, while reasonable, is not the best choice when the variation of spacing is large, and polynomials are not suitable in this case. The better choices are the hyperbolic tangent and the hyperbolic sine. The hyperbolic sine gives a more uniform distribution in the immediate vicinity of the minimum spacing, and thus has less error in this region, but the hyperbolic tangent has the better overall distribution (cf. Section 3 of Chapter 5). These functions are implemented as follows (following Ref. [18]), with the spacing specified at either or both ends, or a point in the interior, of a point distribution on a curve.

Let arc length, s, vary from 0 to 1 as ξ varies from 0 to I: s(0)=0, s(I)=1. Then let the spacing be specified at ξ=0 and ξ=I:

$$s_\xi(0) = \Delta s_1, \qquad s_\xi(I) = \Delta s_2 \tag{48}$$

The hyperbolic tangent distribution is then constructed as follows.

First,

$$A = \frac{\sqrt{\Delta s_2}}{\sqrt{\Delta s_1}} \tag{49}$$

$$B = \frac{1}{I\sqrt{\Delta s_1 \Delta s_2}} \tag{50}$$

Then the following nonlinear equation is solved for δ:

$$\frac{\sinh\delta}{\delta} = B \tag{51}$$

The arc length distribution then is given by

$$s(\xi) = \frac{u(\xi)}{A + (1-A)u(\xi)} \tag{52}$$

where

$$u(\xi) = \frac{1}{2}\left\{1 + \frac{\tanh[\delta(\frac{\xi}{I} - \frac{1}{2})]}{\tanh(\frac{\delta}{2})}\right\} \tag{53}$$

If this is applied to a straight line on which $\underline{r}$ varies from $\underline{r}_0$ to $\underline{r}_I$ we have for the point locations:

$$\underline{r}(\xi) = \underline{r}_0 + (\underline{r}_I - \underline{r}_0)\, s(\xi)$$

The points are then located by taking integer values of ξ:

$$\xi = 0, 1, 2 \ldots, I$$

Clearly the arc length distribution, s(ξ), here is the function ϕ of Eq. (44).

With the spacing hs specified at only ξ=0, the construction proceeds as follows. First B is calculated from

$$B = \frac{1}{I\Delta s} \tag{55}$$

and Eq. (51) is solved for δ. The arc length distribution then is given by

$$s(\xi) = 1 + \frac{\tanh[\frac{\delta}{2}(\frac{\xi}{I} - 1)]}{\tanh(\frac{\delta}{2})} \tag{56}$$

With the spacing specified only at ξ=I the procedure is the same, except that Eq. (56) is replaced by

$$s(\xi) = \frac{\tanh\left(\frac{\delta}{2}\frac{\xi}{I}\right)}{\tanh\left(\frac{\delta}{2}\right)} \tag{57}$$

If the spacing Δs is specified at only an interior point $s=\sigma$, B is again calculated from Eq. (55), and then δ is determined as the solution of

$$1 + \left(\frac{1}{B\sigma\delta}\right)^2 = \left(\frac{\cosh\delta - 1 + \frac{1}{\sigma}}{\sinh\delta}\right)^2 \tag{58}$$

The value of ξ at which $s = \sigma$ is obtained by solving the nonlinear equation

$$X = \frac{I}{\delta}\tanh^{-1}\left(\frac{\sinh\delta}{\frac{1}{\sigma} + \cosh\delta - 1}\right) \tag{59}$$

The arc length distribution then is given by

$$s(\xi) = \sigma\left\{1 + \frac{\sinh\left[\delta\left(\frac{\xi - X}{I}\right)\right]}{\sinh\left(\delta\frac{X}{I}\right)}\right\} \tag{60}$$

This last distribution is based on the hyperbolic sine. From this a distribution baaed on the hyperbolic sine with the spacing specified at one end can be derived. Here B is evaluated from Eq. (55), and then δ is determined as the solution of

$$\frac{\sinh\delta}{\delta} = B \tag{61}$$

The arc length distribution then is given by

$$s(\xi) = \frac{\sinh\left(\delta\frac{\xi}{I}\right)}{\sinh\delta} \tag{62}$$

if the spacing is specified at $\xi=0$. With the specification at $\xi=I$, the distribution is

$$s(\xi) = 1 - \frac{\sinh\left[\delta\left(1 - \frac{\xi}{I}\right)\right]}{\sinh\delta} \tag{63}$$

It is also possible to construct a distribution based on the hyperbolic sine with specified spacing on each end. Here A and B are again calculated from Eq. (49) and (50), but δ is determined from

$$\frac{\tanh(\frac{\delta}{2})}{\frac{\delta}{2}} = B \tag{64}$$

The distribution is then given by Eq. (52), but with

$$u(\xi) = \frac{1}{2}\{1 + \frac{\sinh[\delta(\frac{\xi}{I} - \frac{1}{2})]}{\sinh(\frac{\delta}{2})}\} \tag{65}$$

Finally, a procedure for incorporating the effect of curvature into the distribution function is given in Ref. [38], where the arc length distribution is given in the inverse form by

$$\xi(s) = I\frac{F(s)}{F(s_t)} \tag{66}$$

where

$$F(s) = \int_0^s \sqrt{1 + A(\bar{s})\ K(\bar{s})]^2 + [A'(\bar{s})]^2}\ d\bar{s} \tag{67}$$

s_t is the total arc length, $K(\bar{s})$ is the curvature, and $A(\bar{s})$ is any distribution function (in the inverse form) that would be used without consideration of curvature.

2. Multi-Directional Interpolation

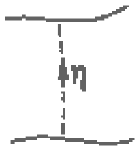
A. Transfinite interpolation

In two directions we may write a linear Lagrange interpolation function individually in each curvilinear direction:

$$\mathbf{r}(\xi,\eta) = \sum_{n=1}^{2} \phi_n(\frac{\xi}{I})\ \mathbf{r}(\xi_n,\eta) \tag{68a}$$

and

$$\mathbf{r}(\xi,\eta) = \sum_{m=1}^{2} \psi_m(\frac{\eta}{J})\ \mathbf{r}(\xi,\eta_m) \tag{68b}$$

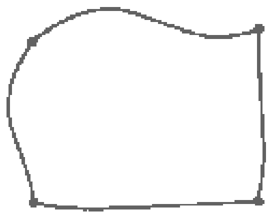


This interpolation is now called "transfinite" since it matches the function on the entire boundary defined by ξ=0 and ξ=I in the first equation, or by η=0 and η=J in the second, i.e., at a nondenumerable number of points, cf. Ref. [40] and [41]).

The tensor product form

$$\underline{r}(\xi,\eta) = \sum_{n=1}^{2} \sum_{m=1}^{2} \phi_n\left(\frac{\xi}{I}\right) \psi_m\left(\frac{\eta}{J}\right) \underline{r}_{nm} \tag{69}$$

where $\underline{r}_{nm} = \underline{r}(\xi_n, \eta_m)$ matches the function at the four corners:



It does not, however, match the function on all the boundary.

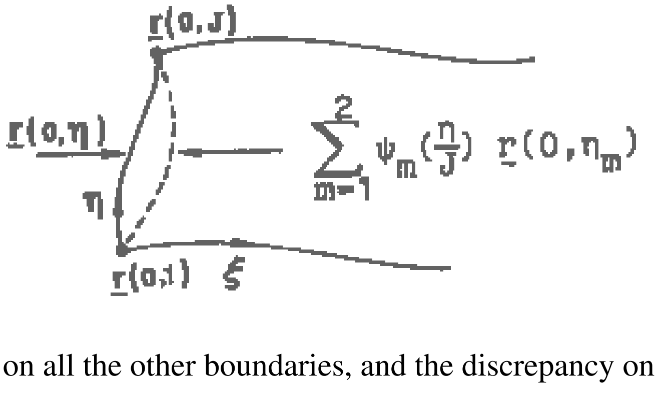
The sum of Eq. (68a) and (68b),

$$\underline{S}(\xi,\eta) = \sum_{n=1}^{2} \phi_n\left(\frac{\xi}{I}\right) \underline{r}(\xi_n,\eta) + \sum_{m=1}^{2} \psi_m\left(\frac{\eta}{J}\right) \underline{r}(\xi,\eta_m) \tag{70}$$

when evaluated on the $\xi=0$ boundary gives

$$\underline{S}(0,\eta) = \underline{r}(0,\eta) + \sum_{m=1}^{2} \psi_m\left(\frac{\eta}{J}\right) \underline{r}(0,\eta_m) \tag{71}$$

This does not match the function on the $\xi=0$ boundary because of the second term on the right, which is an interpolation between the ends of this boundary:



Similar effects occur on all the other boundaries, and the discrepancy on the $\xi=I$ boundary is

$$\sum_{m=1}^{2} \psi_m\left(\frac{\eta}{J}\right) \underset{\sim}{r}(I,\eta_m)$$

The discrepancies on both of these boundaries can be removed by subtracting from $\underset{\sim}{S}(\xi,\eta)$ a function formed by interpolating the discrepancies between the two boundaries:

$$\underline{R}(\xi,\eta) = \sum_{n=1}^{2} \phi_n\left(\frac{\xi}{I}\right) \left[\sum_{m=1}^{2} \psi_m\left(\frac{\eta}{J}\right) \underline{r}(\xi_n,\eta_m)\right] \tag{72}$$

But this is simply the tensor product form given by Eq. (69), which matches the function at the four corners.

The function $\underset{\sim}{S}-\underset{\sim}{R}$ then matches the function on all four sides of the boundary, so that we have the transfinite interpolation form,

$$\begin{aligned} \underset{\sim}{r}(\xi,\eta) = {} & \sum_{n=1}^{2} \phi_n\left(\frac{\xi}{I}\right) \underline{r}(\xi_n,\eta) + \sum_{m=1}^{2} \psi_m\left(\frac{\eta}{J}\right) \underset{\sim}{r}(\xi,\eta_m) \\ & - \sum_{n=1}^{2} \sum_{m=1}^{2} \phi_n\left(\frac{\xi}{I}\right) \psi_m\left(\frac{\eta}{J}\right) \underset{\sim}{r}(\xi_n,\eta_m) \end{aligned} \tag{73}$$

which matches the function on the entire boundary. By contrast, the tensor product form

$$\underline{r}(\xi,\eta) = \sum_{n=1}^{2} \sum_{m=1}^{2} \phi_n\left(\frac{\xi}{I}\right) \psi_m\left(\frac{\eta}{J}\right) \underline{r}(\xi_n,\eta_m) \tag{74}$$

matches the function only at the four corners on the boundary. This generalizes to the interpolation from a set of N+M intersecting curves for which the univariate interpolation is given by

$$\underset{\sim}{r}(\xi,\eta) = \sum_{n=1}^{N} \phi_n\left(\frac{\xi}{I}\right) \underset{\sim}{r}(\xi_n,\eta) \tag{75a}$$

and

$$\underset{\sim}{r}(\xi,\eta) = \sum_{m=1}^{M} \psi_m\left(\frac{\eta}{J}\right) \underset{\sim}{r}(\xi,\eta_m) \tag{75b}$$

where now the "blending" functions, ϕ_n and ψ_m, are any functions which satisfy the cardinality conditions

$$\phi_n\left(\frac{\xi_l}{I}\right) = \delta_{nl} \qquad n = 1,2,\ldots,N, \qquad l = 1,2,\ldots,N$$

$$\psi_m\left(\frac{\eta_l}{J}\right) = \delta_{ml} \qquad m = 1,2,\ldots,M, \qquad l = 1,2,\ldots,M \tag{76}$$

The general form of the transfinite interpolation then is

$$\underset{\sim}{r}(\xi,\eta) = \sum_{n=1}^{N} \phi_n\left(\frac{\xi}{I}\right) \underset{\sim}{r}(\xi_n,\eta) + \sum_{m=1}^{M} \psi_m\left(\frac{\eta}{J}\right) \underset{\sim}{r}(\xi,\eta_m)$$

$$- \sum_{n=1}^{N} \sum_{m=1}^{M} \phi_n\left(\frac{\xi}{I}\right) \psi_m\left(\frac{\eta}{J}\right) \underset{\sim}{r}(\xi_n,\eta_m) \tag{77}$$

while the tensor-product form is

$$\underset{\sim}{r}(\xi,\eta) = \sum_{n=1}^{N} \sum_{m=1}^{M} \phi_n\left(\frac{\xi}{I}\right) \psi_m\left(\frac{\eta}{J}\right) \underset{\sim}{r}(\xi_n,\eta_m) \tag{78}$$

Eq. (77) can be written in the form

$$\underset{\sim}{r}(\xi,\eta) = \sum_{m=1}^{M} \psi_m\left(\frac{\eta}{J}\right)\underset{\sim}{r}(\xi,\eta_m) +$$

$$\sum_{n=1}^{N} \phi_n\left(\frac{\xi}{I}\right) \left[\underset{\sim}{r}(\xi_n,\eta) - \sum_{m=1}^{M} \psi_m\left(\frac{\eta}{J}\right) \underset{\sim}{r}(\xi_n,\eta_m)\right]$$

But here the first term is the result at each point in the field of the unidirectional interpolation in the direction, and the bracket is the difference between the specified values on the $\xi=\xi_n$ lines and the result of the unidirectional interpolation on those lines. The two-directional transfinite interpolation can thus be implemented in two unidirectional interpolation steps by first performing the unidirectional interpolation in one direction, say η, over the entire field, calling the result $\underset{\sim}{F}_1(\xi,\eta)$:

$$\underset{\sim}{F}_1(\xi,\eta) = \sum_{m=1}^{M} \psi_m\left(\frac{\eta}{J}\right) \underset{\sim}{r}(\xi,\eta_m) \tag{79a}$$

then interopolating the discrepancy on the $\xi=\xi_n$ lines over the entire field in the other direction, ξ here, calling the result $\underset{\sim}{F}_2(\xi,\eta)$:

$$\underset{\sim}{F}_2(\xi,\eta) = \sum_{n=1}^{N} \phi_n\left(\frac{\xi}{I}\right)\left[\underset{\sim}{r}(\xi_n,\eta) - \underset{\sim}{F}_1(\xi_n,\eta)\right] \tag{79b}$$

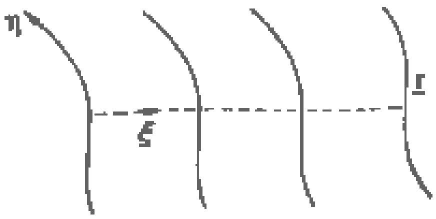
and then adding $\mathbf{F}_1$ and $\mathbf{F}_2$.

$$\mathbf{r}(\xi,\eta) = \mathbf{F}_1(\xi,\eta) + \mathbf{F}_2(\xi,\eta) \tag{79c}$$

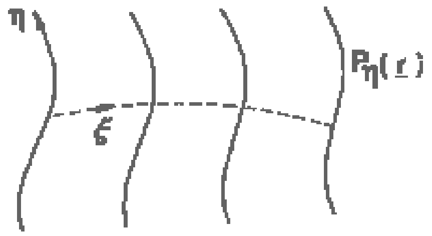
The transfinite interpolation form given by Eq. (77) is the algebraically best approximation, while the tensor product from of Eq. (78) is the algebraically worst (cf. Ref. [40]). The difference between these two forms should be fully understood. The transfinite interpolation form, Eq. (77), interpolates to the entirety of a set of intersecting arbitrary curves, while the tensor product form, Eq. (78), interpolates only to the intersections of these curves. The interpolation function defined by Eq. (77) with N=M=2, using the Lagrange interpolation polynomials as the blending functions, is termed the transfinite bilinear interpolant. With N=M=3, this form is the transfinite biquainterpolant. Other immediate candidates for the blending functions are the Hermite interpolation polynomials and the splines, since these all can be expressed in the form of Eq. (75). The spline-blended form gives the smoothest grid with continuous second derivatives.

B. Projectors

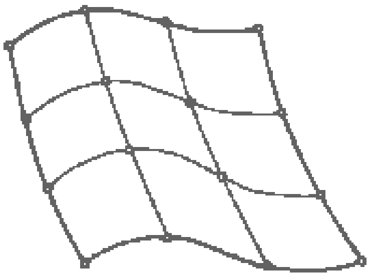
Now let $P_\xi(\mathbf{r})$ be a one-dimensional interpolation function in the ξ-direction which matches $\mathbf{r}$ on the N lines, $\xi = \xi_n$ (n=1,2,...N),:



(Note that the subscript ξ here does not denote differentiation.) Similarly, let $P_\eta(\mathbf{r})$ match $\mathbf{r}$ on the M lines, $\eta = \eta_m$ (m=1,2,...M). These interpolations are performed by projectors, P_ξ and P_η, which are assumed to be idempotent linear n operators. Protectors are discussed in more detail in Ref. [40]. Some discussion is also given in Ref. [37]. The product projector, $P_\xi[P_\eta(\mathbf{r})]$, then matches the function $P_\eta(\mathbf{r})$, instead of $\mathbf{r}$, on the N lines, $\xi = \xi_n$:



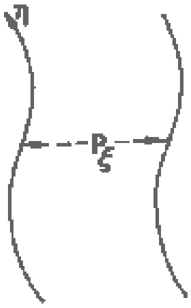
Then, since $P_\eta(\mathbf{r})$ matches $\mathbf{r}$ on the M lines, $\eta = \eta_m$, it follows that the product projector will match $\underset{\sim}{r}$ at the NxM points (ξ_n, η_m):

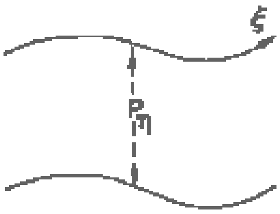


Clearly the same conclusion is reached for the product projector $P_\eta[P_\xi(\underset{\sim}{r})]$, so that the protectors P_ξ and P_η commute.

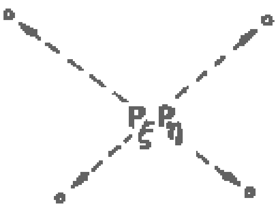
The sum projector, $P_\xi(\underset{\sim}{r})+P_\eta(\underset{\sim}{r})$ matches $\underset{\sim}{r}+P_\eta(\underset{\sim}{r})$ on the N lines $\xi=\xi_n$, and matches $\underset{\sim}{r}+P_\xi(\underset{\sim}{r})$ on the M lines $\eta=\eta_m$. It should be clear then that the projector, $P_\xi(\underset{\sim}{r})+P_\eta(\underset{\sim}{r})-P_\xi[P_\eta(\underset{\sim}{r})]$ will match $\underset{\sim}{r}$ on the N lines $\xi=\xi_n$, since $P_\xi[P_\eta(\underset{\sim}{r})]$ matches $P_\eta(\underset{\sim}{r})$ on these lines. Similarly, the projector $P_\xi(\underset{\sim}{r})+P_\eta(\underset{\sim}{r})-P_\eta[P_\xi(\underset{\sim}{r})]$ matches $\underset{\sim}{r}$ on the M lines $\eta=\eta_m$. Therefore, since $P_\xi + P_\eta = P_\eta P_\xi$, the Boolean sum projector, $P_\xi \oplus P_\eta = P_\xi + P_\eta - P_\xi P_\eta$, will match $\underset{\sim}{r}$ on the entirety of the N+M lines $\xi=\xi_n$ and $\eta=\eta_m$ which includes, of course, the entire boundary of the region.

In summary, the individual protectors, P_ξ and P_η, interpolate undirectionally between two opposing boundaries:

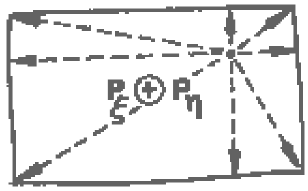




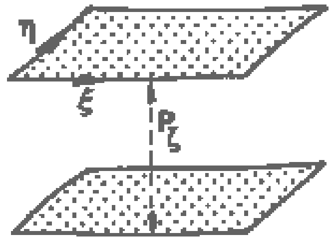
The product progeotor, $P_\xi P_\eta$, interpolates in two directions from the four corners:



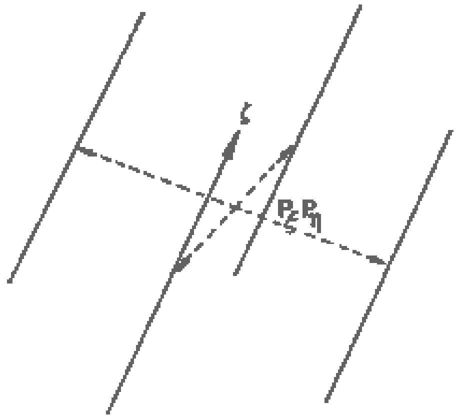
The Boolean sum projector, $P_\xi \oplus P_\eta$, interpolates from the entire boundary:



In three dimensions, the individual protectors, P_ξ, P_η, and P_ζ, interpolate undirectionally between two opposing faces of the six-sided region:



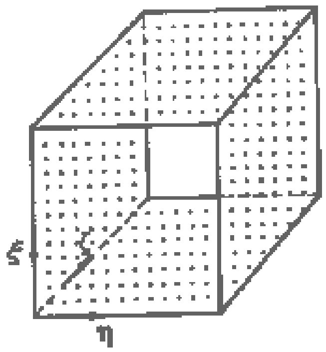
(matching $\underset{\sim}{r}$ on each of the two faces in each case). The double product projector, $P_\xi P_\eta$, interpolates in two directions from the four edges along which ξ and η are constant:



(matching $\underset{\sim}{r}$ on each of these edges). The Boolean sum projector

$$P_\xi \oplus P_\eta = P_\xi + P_\eta - P_\xi P_\eta \tag{80}$$

interpolates in two directions from the four faces on which either ξ or η is constant:

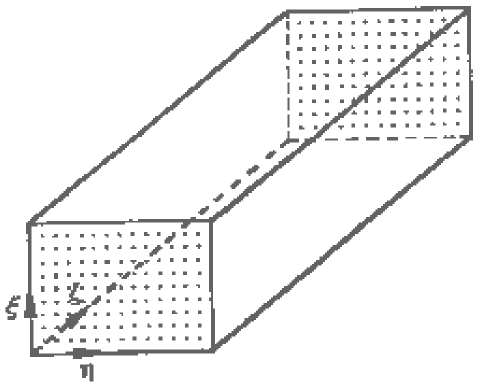


(matching $\underset{\sim}{r}$ on all of these faces).

The Boolean sum projector

$$P_\xi P_\eta \oplus P_\zeta = P_\xi P_\eta + P_\zeta - P_\xi P_\eta P_\zeta \tag{81}$$

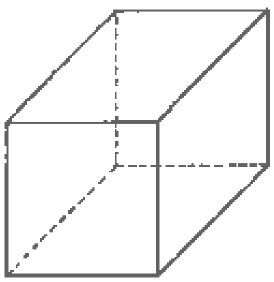
interpolates in three directions, matching $\underset{\sim}{r}$ on the four edges on which ξ and η are constant and also on the two faces on which ζ is constant:



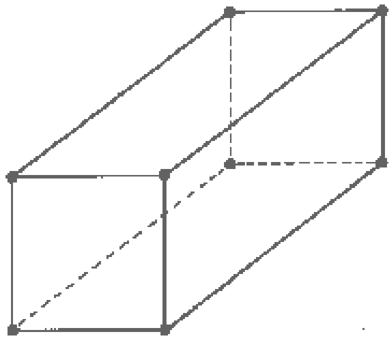
The Boolean sum projector

$$P_\xi P_\eta \oplus P_\eta P_\zeta \oplus P_\zeta P_\xi = P_\xi P_\eta + P_\eta P_\zeta + P_\zeta P_\xi - 2P_\xi P_\eta P_\zeta \tag{82}$$

interpolates in three directions, with $\underset{\sim}{r}$ matched on all twelve edges:



The triple product projector, $\xi \eta \zeta$, interpolates $\underset{\sim}{r}$ from the eight corners:



Finally the Boolean sum projector

$$P_\xi \oplus P_\eta \oplus P_\zeta = P_\xi + P_\eta + P_\zeta - P_\xi P_\eta - P_\eta P_\zeta - P_\zeta P_\xi + P_\xi P_\eta P_\zeta \qquad (83)$$

matches $\underset{\sim}{r}$ on the entire boundary.

Much cancellation occurs in the algebraic manipulation of the projectors involved in developing the above relations, since $P_\xi P_\xi = P_\xi$, etc. Thus, for example,

$$P_\xi \oplus P_\xi P_\eta = P_\xi + P_\xi P_\eta - P_\xi P_\xi P_\eta = P_\xi + P_\xi P_\eta - P_\xi P_\eta = P_\xi$$

This is to be expected since interpolation by P_ξ matches the function on all of the two sides on which ξ is constant, while $P_\xi P_\eta$ matches the function on the four edges on which ξ and η are constant. But these edges are contained on the two sides cited, so that nothing is changed by adding $P_\xi P_\eta$ to P_ξ in the Boolean sense. The projector formed as the Boolean sum of all three of the individual projectors is algebraically maximum, while the triple product projector is algebraically minimal.

The importance of the projectors is that the structure given above allows multi-directional interpolation to be constructed systematically from unidirectional forms. With one-dimensional interpolation of the form of Eq. (75) we have

$$P_\xi(\underset{\sim}{r}) = \sum_{n=1}^{N} \phi_n\left(\frac{\xi}{I}\right) \underset{\sim}{r}(\xi_n, \eta) \qquad (84a)$$

$$P_\eta(\underset{\sim}{r}) = \sum_{m=1}^{M} \psi_m\left(\frac{\eta}{J}\right) \underset{\sim}{r}(\xi, \eta_m) \qquad (84b)$$

so that

$$P_\xi P_\eta(\underline{r}) = P_\xi[P_\eta(\underline{r})]$$

$$= \sum_{n=1}^{N} \phi_n\left(\frac{\xi}{I}\right) \left[\sum_{m=1}^{M} \psi_m\left(\frac{\eta}{J}\right) \underline{r}(\xi_n, \eta_m)\right] \tag{85}$$

$$= \sum_{n=1}^{N} \sum_{m=1}^{M} \phi_n\left(\frac{\xi}{I}\right) \psi_m\left(\frac{\eta}{J}\right) \underline{r}(\xi_n, \eta_m)$$

which is just the tensor product form given previously in Eq. (78), so that the two-directional transfinite interpolation corresponding to the projector $P_\xi \oplus P_\eta$ is just that given by Eq. (77). As noted above, spline interpolation also falls directly into this form, so that the multi-directional transfinite interpolation based on splines requires only the determination of the splines separately in the individual directions.

Although Hermite interpolation can be defined in terms of additional points, and thus be put in this same form also, the use of projectors allows a more direct statement as follows. For the projectors we have, following Eq. (5),

$$P_\xi(\underline{r}) = \sum_{n=1}^{N} \phi_n\left(\frac{\xi}{I}\right) \underline{r}(\xi_n, \eta) + \sum_{n=1}^{N} \phi_n\left(\frac{\xi}{I}\right) \underline{r}_\xi(\xi_n, \eta) \tag{86a}$$

and

$$P_\eta(\underline{r}) = \sum_{m=1}^{M} \psi_m\left(\frac{\eta}{J}\right) \underline{r}(\xi, \eta_m) + \sum_{m=1}^{M} \psi_m\left(\frac{\eta}{J}\right) \underline{r}_\eta(\xi, \eta_m) \tag{86b}$$

Now

$$P_\xi P_\eta(\underline{r}) = P_\xi[P_\eta(\underline{r})]$$

$$= \sum_{n=1}^{N} \phi_n\left(\frac{\xi}{I}\right) \left[\sum_{m=1}^{M} \psi_m\left(\frac{\eta}{J}\right) \underline{r}(\xi_n, \eta_m)\right.$$

$$\left.+ \sum_{m=1}^{M} \Psi_m\left(\frac{\eta}{J}\right) \underline{r}_\eta(\xi_n, \eta_m)\right]$$

$$+ \sum_{n=1}^{N} \Phi_n\left(\frac{\xi}{I}\right) \left[\sum_{m=1}^{M} \psi_m\left(\frac{\eta}{J}\right) \underline{r}_\xi(\xi_n, \eta_m)\right. \tag{87}$$

$$\left.+ \sum_{m=1}^{M} \Psi_m\left(\frac{\eta}{J}\right) \underline{r}_{\xi\eta}(\xi_n, \eta_m)\right]$$

$$= \sum_{n=1}^{N} \sum_{m=1}^{M} \left[\phi_n\left(\frac{\xi}{I}\right) \psi_m\left(\frac{\eta}{J}\right) \underline{r}(\xi_n, \eta_m)\right.$$

$$+ \phi_n\left(\frac{\xi}{I}\right) \Psi_m\left(\frac{\eta}{J}\right) \underline{r}_\eta(\xi_n, \eta_m) + \Phi_n\left(\frac{\xi}{I}\right) \psi_m\left(\frac{\eta}{J}\right) \underline{r}_\xi(\xi_n, \eta_m)$$

$$\left.+ \Phi_n\left(\frac{\xi}{I}\right) \Psi_m\left(\frac{\eta}{J}\right) \underline{r}_{\xi\eta}(\xi_n, \eta_m)\right]$$

Then the two-directional transfinite interpolation can be constructed by substitution of Eq. (86) and (87) into the projector $P_\xi \oplus P_\eta$. Here the tensor product form, $P_\xi P_\eta$, interpolates from the values of the function, its two first derivatives, and the cross-derivative at the four corners of the boundary. The transfinite interpolation form, $P_\xi \oplus P_\eta$, however, interpolates from the value of the function and its normal derivative on the entire boundary.

The triple product corresponding to Eq. (84) is simply

$$P_\xi P_\eta P_\zeta(\underline{r}) = \sum_{n=1}^{N} \sum_{m=1}^{M} \sum_{l=1}^{L} \phi_n\left(\frac{\xi}{I}\right) \psi_m\left(\frac{\eta}{J}\right) \theta_l\left(\frac{\zeta}{K}\right) \underline{r}(\xi_n, \eta_m, \zeta_l) \tag{88}$$

Recall that with the unidirectional form given by Eq. (44), we have in these relations L=M=N=2 and

$$\phi_1 = 1 - \phi, \qquad \phi_2 = \phi$$

$$\psi_1 = 1 - \psi, \qquad \psi_2 = \psi$$

$$\theta_1 = 1 - \theta, \qquad \theta_2 = \theta$$

The above evaluations of the product projectors serve to illustrate the evaluation of such products for general projectors, i.e., that the effect of the product protectors is simply an interpolation in one-direction of an interpolant in another direction. This allows the multi-directional transfinite interpolation to be constructed from the Boolean sums of the protectors given above using any appropriate unidirectional interpolation forms as the basis projectors. It should also be noted that the unidirectional interpolation does not have to be of the same form in all the directions. Thus Lagrange interpolation could be used in one of the directions while Hermite is used in the other direction of a two-directional construction. As noted above, the blending functions do not have to be polynomials. In fact, all of the unidirectional interpolation that was discussed earlier in this chapter can be applied in the context of multi-directional interpolation based on the protectors. This freedom to combine different types of univariate interpolation gives considerable flexibility to transfinite interpolation based on the projector structure, and allows attention to be focused on developing appropriate unidirectional interpolations, the multi-directional format then following automatically.

The protectors allow the transfinite interpolation to be easily set up as a sequence of unidirectional interpolations, in the manner discussed above. Thus in the two-directional case, Eq. (80) can be written as

$$P_\xi \oplus P_\eta = P_\xi + P_\eta \,(I - P_\xi) \tag{89}$$

where I indicates the identity operation. But here the first term is clearly the unidirectional interpolation in the ξ-direction, while the parenthesis (I-P ξ) is the discrepancy on the η-lines on which $\underset{\sim}{r}$ is specified that results from this ξ-interpolation. The second term then is the unidirectional interpolation of this descrepancy interpolated in the η-direction.

The two-directional interpolation thus can be implemented by: (1) interpolating $\underset{\sim}{r}$ in the ξ-direction, (2) calculating the discrepancy between $\underset{\sim}{r}$ and this result on the η-lines that are to be used in the η-interpolation, (3) interpolating this discrepancy in the η-direction, and (4) adding the result of this ξ-interpolation to that of the interpolation. Symbolically these steps can be stated as the following:

$$\begin{aligned} \underset{\sim}{r}_1 &= P_\xi \, \underset{\sim}{r} \\ \underset{\sim}{r}_2 &= P_\eta \,(\underset{\sim}{r} - \underset{\sim}{r}_1) \\ \underset{\sim}{r}(\xi,\eta) &= \underset{\sim}{r}_1 + \underset{\sim}{r}_2 \end{aligned} \tag{90}$$

Obviously, the order of the unidirectional interpolation is immaterial.

Similarly, Eq. (83) can be written as

$$P_\xi \oplus P_\eta \oplus P_\zeta = P_\xi + (P_\eta + P_\zeta - P_\eta P_\zeta)(I - P_\xi) \tag{91}$$

The three-directional interpolation thus can be implemented by (1) interpolating $\underset{\sim}{r}$ in the ξ-direction, (2) calculating the discrepancy between $\underset{\sim}{r}$ and this result on the η-surfaces and ζ-surfaces that are to be used in the interpolation in those directions, (3) interpolating this discrepancy by two-directional interpolation, and (4) adding this result to that of the ξ-interpolation. These operations can be stated as

$$\begin{aligned} \underset{\sim}{F}_1 &= P_\xi \underset{\sim}{r} \\ \underset{\sim}{F}_2 &= P_\eta (\underset{\sim}{r} - \underset{\sim}{F}_1) \\ \underset{\sim}{F}_3 &= P_\zeta (\underset{\sim}{r} - \underset{\sim}{F}_1 - \underset{\sim}{F}_2) \\ \underset{\sim}{r}(\xi,\eta) &= \underset{\sim}{F}_1 + \underset{\sim}{F}_2 + \underset{\sim}{F}_3 \end{aligned} \tag{92}$$

Exercises

1. Show that with N=2 and ψ_n constant, the multi-surface interpolation is equivalent to the linear Lagrange interpolation. Mote that the Lagrange polynomials here satisfy Eq. (25) with $\frac{\xi_2}{I} = \frac{1}{2}$. For other choices of ξ_2, other quadratic polynomials result from Eq. (25), so that there exists a one-parameter family of cubic forms of the multi-surface interpolation. Similarly, a two-parameter family of quadratic forms exists, etc.

2. Show that the quadratic form of the multi-surface interpolation is given by

$$\underset{\sim}{r}(\xi) = \left(1 - \frac{\xi}{I}\right)^2 \underset{\sim}{r}(0) + 2\frac{\xi}{I}\left(1 - \frac{\xi}{I}\right)\underset{\sim}{F}_2 + \left(\frac{\xi}{I}\right)^2 \underset{\sim}{r}(I)$$

3. Show that the quadratic forms of the multi-surface and Bezier interpolations are equivalent.

4. Show that with N=4 and ψ_n the quadradic Lagrange interpolation polynomials given on p. 282, the interpolation functions for the multi-surface interpolation are given by

$$G_1(\xi) = \frac{\xi}{I} - \frac{3}{2}\left(\frac{\xi}{I}\right)^2 + \frac{2}{3}\left(\frac{\xi}{I}\right)^3$$

$$G_2(\xi) = 2\left(\frac{\xi}{I}\right)^2 - \frac{4}{3}\left(\frac{\xi}{I}\right)^3$$

$$G_3(\xi) = \frac{2}{3}\left(\frac{\xi}{I}\right)^3 - \frac{1}{2}\left(\frac{\xi}{I}\right)^2$$

the multi-surface interpolation is equivalent to the cubic Hermite interpolation.

5. Show that $S(\xi)$ is given by Eq. (38) for Lagrange interpolation. (Hint: If all the $\underset{\sim}{r}_n$ in Eq. (2) are the same, the interpolation must reproduce this value, hence the Lagrange interpolation polynomials satisfy

$$\sum_{n=1}^{N} \phi_n\left(\frac{\xi}{I}\right) = 1$$

6. Show that for quadratic Lagrange interpolation, uniformity requires that $\underset{\sim}{r}_2$ be selected such that

$$[r_2 - r(0)] \cdot I = \frac{1}{2}$$

Note that this does not completely determine $\underset{\sim}{r}_2$.

7. Show that uniformity is achieved with cubic Hermite interpolation with $\frac{\xi_2}{I} = \frac{1}{2}$ with orthogonality at the boundaries if the spacings at the boundaries are given by

$$s = \frac{1}{n \cdot I}$$

where $\underset{\sim}{n}$ is the unit normal to the boundary. This completely specifies the interpolation in this case. However, as noted in Exercise 1, ξ_2 is a free parameter.

8. Show that for multi-surface interpolation with N=4 and $\frac{\xi_2}{I} = \frac{1}{2}$ uniformity is achieved with

$$[r_2 - r(0)] \cdot I = [r(I) - r_3] \cdot I = \frac{1}{6}$$

9. Show also that with orthogonality at the boundary the result of Exercise 8 completely determines all of the interpolation parameters, i.e., that

$$P_2 = r(0) + \frac{1}{6n(0)\cdot I}\, n(0)$$

$$P_3 = r(I) - \frac{1}{6n(I)\cdot I}\, n(I)$$

where n is the unit normal to the boundary. Hint: Use Eq. (7) and (29). For general ξ_2 the 1/6 is replaced by 1/2 - 1/[6(ξ_2/I)] and 1/2 - 1/[6(1-ξ_2/I)] in the above expressions involving P_2 and P_3, respectively. Some effects of the choice of ξ_2 are shown in Ref. [32].

10. Show that local uniformity on the interval $\xi_n \leq \xi \leq \xi_{n+1}$ for the multi-surface interpolation based on piecewise-linear functions requires that

$$\frac{C_n}{C_{n+1}} = \frac{\xi_{n+1} - \xi_{n-1}}{\xi_{n+2} - \xi_n}$$

where $C_n = (\overline{P}_{n+1} - \overline{P}_n)\cdot I$, with I

11. Consider a rectangular physical region with equally-spaced points on the bottom and top, but with unequal spacing on the left and right sides (but with the same point distribution on both of these sides). Show that horizontal interpolation will reflect the unequal spacing of the horizontal grid lines in the field, but that vertical interpolation will not. Show also that the unequal spacing is reflected with transfinite interpolation.

12. Show that transfinite interpolation based on linear blending functions will reflect the unequal boundary point spacing in the field for the rectangular physical region of Exercise 11, but will not for a C-grid. From the consideration of transfinite interpolation as a sequence of unidirectional interpolations, explain why this is so.

13. Show that with cubic Lagrange interpolation the locations of the two intermediate surfaces, r_2 and r_3, are related to the slopes at both ends. Note the contrast between this and the multi-surface interpolation where each of the intermediate surfaces depends on only the slope at one end.

14. Give the cubic form of Lagrange interpolation.

15. Show that in two dimensions transfinite interpolation is equivalent to a generation system based on the fourth - order partial differential equation

$$r_{\xi\xi\eta\eta} = 0$$

(This is also equivalent to the quadralaterial isoparemetric elements often used to construct finite element meshes.)