APPENDIX A

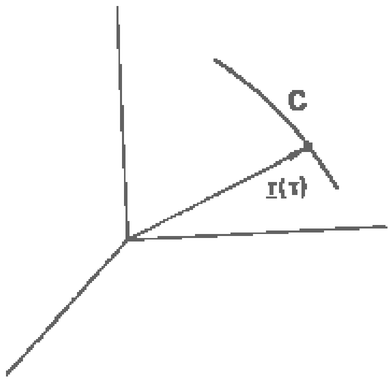
DIFFERENTIAL-GEOMETRIC CONCEPTS ON SPACE CURVES AND SURFACES

1. Theory of Curves

In this appendix we consider only those parts of the theory of curves in space which are needed in the theory of surface geometry for the purpose of coordinate generation. Let C be a curve in space whose parametric equation is given as

$$\mathbf{r} = \mathbf{r}(\tau)$$

where τ is a parameter which takes values in a certain interval a $\leq \tau \leq$ b.



It is assumed that the real vector function $\underset{\sim}{r}(\tau)$ is p $\geq$ 1 times continuously differentiable for all values of τ in the specified interval, and at least one component of the first derivative

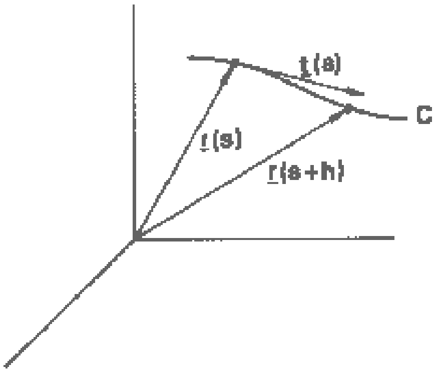
$$\mathbf{r}' = \frac{d\mathbf{r}}{d\tau}$$

is different from zero. Note that the parameter τ can be replaced by some other parameter, say s, provided that ds/d$\tau \neq$ 0.

A. Tangent vector

Let us consider the arc length s as a parameter. Then the coordinates of two neighboring points on the curve are $\underset{\sim}{r}$(s) and $\underset{\sim}{r}$(s+h). The vector $\underset{\sim}{t}$(s) defined as

$$\mathbf{t}(s) = \lim_{h \to 0} \frac{\mathbf{r}(s+h) - \mathbf{r}(s)}{h} = \frac{d\mathbf{r}}{ds} \tag{3}$$



is the unit tangent vector at the point s on the curve. Since $|d\underline{r}| = ds$, we immediately see that $|\underline{t}(s)| = 1$.

If the curve C is referred to a general coordinate system ξ^i, then its parametric equations are given as

$$\xi^i = \xi^i(s), \qquad i = 1,2,3$$

In this case, using the chain rule of differentiation, we can write

$$\underline{t}(s) = \sum_{i=1}^{3} \underline{a}_i \frac{d\xi^i}{ds} \tag{4}$$

where $\underline{a}_i$ are the covariant base vectors defined in Eq. (III-1).

B. Principal normal

Since $\underline{t} \cdot \underline{t} = 1$, a single differentiation with respect to s yields

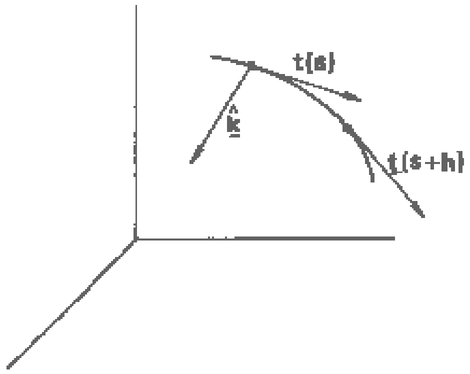
$$\underline{t} \cdot \frac{d\underline{t}}{ds} = 0$$

so that the vector $d\underline{t}/ds$ is orthogonal to $\underline{t}$. The vector

$$\hat{\underline{k}} = \frac{d\underline{t}}{ds} \tag{5}$$

is called the curvature vector. The unit principal normal vector is then defined as

$$\underline{p} = \hat{\underline{k}}/|\hat{\underline{k}}| \tag{6}$$



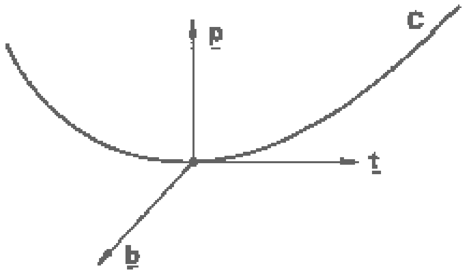
The magnitude $k(s) = \left|\hat{\underset{\sim}{k}}\right|$ and its reciprocal ρ = 1/k(s) are, respectively, the curvature and the radius of curvature of the curve at the point under consideration. Both the curvature vector and the principal normal are directed toward the center of curvature of the curve at that point.

C. Normal and osculating planes

The totality of all vectors which are bound at a point of the curve and which are orthogonal to the unit tangent vector at that point lie in a plane. This plane is called the normal plane. The plane formed by the unit tangent and the principal normal vector is called the osculating plane.

D. Binormal vector

A unit vector $\underset{\sim}{b}$ (s) which is orthogonal to both $\underset{\sim}{t}$ and $\underset{\sim}{p}$ is called the binormal vector. Its orientation is fixed by taking $\underset{\sim}{t}$, $\underset{\sim}{p}$, $\underset{\sim}{b}$ to form a right-handed triad as shown below:



$$\underline{b} = \underline{t} \times \underline{p} \tag{7}$$

Note that for plane curves the binormal $\underset{\sim}{b}$ is the constant unit vector normal to the plane, and the principal normal is the usual normal to the curve directed toward the center of curvature at that point.

The twisted curves in space have their binormals as functions of s. Because of twisting a new quantity called torsion appears, which is obtained as follows. Consider the obvious equations

$$\mathbf{b} \cdot \mathbf{b} = 1, \quad \mathbf{b} \cdot \mathbf{t} = 0 \tag{8}$$

Differentiating each equation with respect to s, we obtain

$$\mathbf{b} \cdot \frac{d\mathbf{b}}{ds} = 0 \tag{9a}$$

$$\mathbf{b} \cdot \frac{d\mathbf{t}}{ds} + \frac{d\mathbf{b}}{ds} \cdot \mathbf{t} = 0 \tag{9b}$$

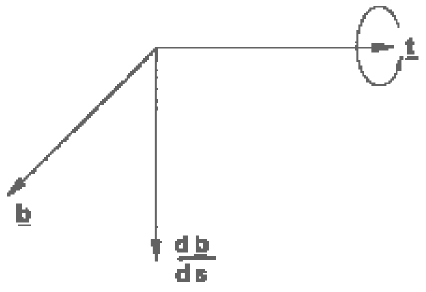
Thus

$$\frac{d\mathbf{b}}{ds} \cdot \mathbf{t} = -k\mathbf{b} \cdot \mathbf{p} = 0 \tag{9c}$$

From (9a,c) we find that $d\mathbf{b}/ds$ is a vector which is orthogonal to both $\mathbf{t}$ and $\mathbf{b}$. Thus $d\mathbf{b}/ds$ lies along the principal normal,

$$\frac{d\mathbf{b}}{ds} = \pm \tau \mathbf{p}$$

To decide about the sign we take the cross product of $\mathbf{b}$ with $d\mathbf{b}/ds$ and take it as a positive rotation about $\mathbf{t}$:



Thus

$$\mathbf{b} \times \frac{d\mathbf{b}}{ds} = \tau \mathbf{t} \tag{10a}$$

and

$$\frac{d\mathbf{b}}{ds} = -\tau \mathbf{p} \tag{10b}$$

E. Serret-Frenet equations

A set of equations known as the Serret-Frenet equations, which are the intrinsic equations of a curve, are the following. Differentiating the equation

$$\underset{\sim}{p} = \underset{\sim}{b} \times \underset{\sim}{t}$$

with respect to s, we have

$$\frac{d\underset{\sim}{p}}{ds} = \tau \underset{\sim}{b} - k \underset{\sim}{t} \qquad (11)$$

Equations (6), (10) and (11) are the Serret-Frenet equations, and are collected below:

$$\frac{d\underset{\sim}{t}}{ds} = k\underset{\sim}{p}\,, \qquad k = \text{curvature} \qquad (12a)$$

$$\frac{d\underset{\sim}{b}}{ds} = -\tau\underset{\sim}{p}\,, \qquad \tau = \text{torsion} \qquad (12b)$$

$$\frac{d\underset{\sim}{p}}{ds} = \tau \underset{\sim}{b} - k \underset{\sim}{t} \qquad (12c)$$

For a plane curve, $\tau = 0$, so that

$$\underset{\sim}{b} = \text{constant}$$

$$\frac{d\underset{\sim}{t}}{ds} = k\underset{\sim}{p}, \qquad \frac{d\underset{\sim}{p}}{ds} = -k\underset{\sim}{t} \qquad (13)$$

2. Geometry of Two-Dimensional Surfaces Embedded in E^3

Before taking up the main subject of surface theory, it is important to clarify the notations which are to be used in the ensuing development.

In an Euclidean E^3, a set of rectangular cartesian coordinates (x,y,z) can always be introduced. As before, in E^3 a general curvilinear coordinate system will be denoted by ξ^i (i = 1,2,3). With these curvilinear coordinates, a surface in E^3 will be denoted by ξ^{ν} = constant, where ν = 1,2,3. The following convention is adopted which maintains the right-handedness of the two remaining current coordinates: On the surface ξ^{ν} = constant, the current coordinates are ξ^{α}, ξ^{β}, where (ν,α,β) are cyclic.

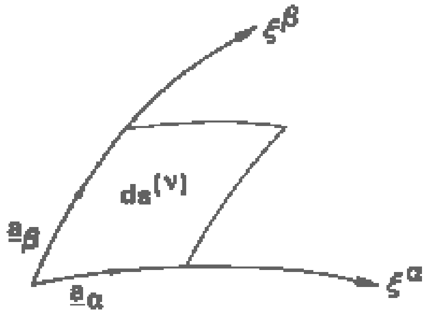
A. First fundamental form

Let us consider the surface ξ^{ν} = constant. In this surface an element of length $ds^{(\nu)}$ is then given by

$$(ds^{(\nu)})^2 = d\mathbf{r} \cdot d\mathbf{r}$$

$$= \sum_{\alpha,\beta} \frac{\partial \mathbf{r}}{\partial \xi^\alpha} \cdot \frac{\partial \mathbf{r}}{\partial \xi^\beta} \, d\xi^\alpha d\xi^\beta \tag{14}$$

$$= \sum_{\alpha,\beta} \mathbf{a}_\alpha \cdot \mathbf{a}_\beta \, d\xi^\alpha d\xi^\beta$$



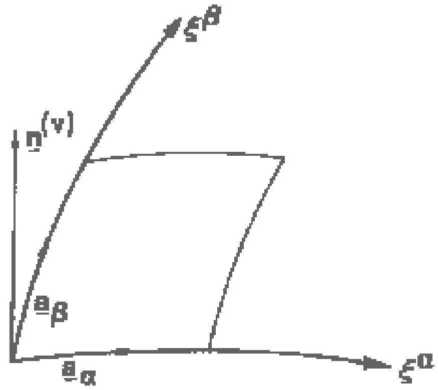
where the indices α and β will assume only the two values different from ν. Eq. (14) is called the first fundamental form of a surface.

B. Unit normal vector

The unit normal to the surface ξ^ν = constant is defined as

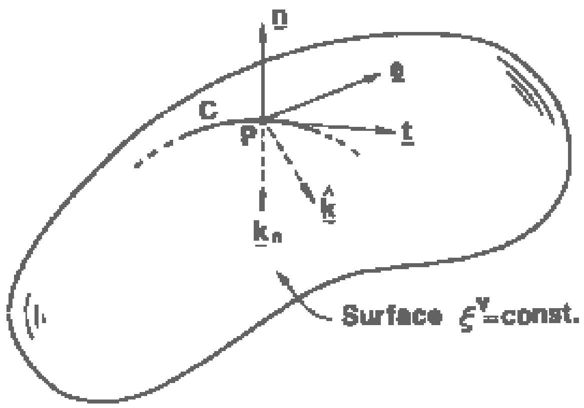
$$\mathbf{n}^{(\nu)} = \frac{1}{\sqrt{gg^{\nu\nu}}} \frac{\mathbf{a}_\alpha \times \mathbf{a}_\beta}{|\mathbf{a}_\alpha \times \mathbf{a}_\beta|}, \tag{15}$$

where again (ν,α,β) are cyclic.



C. Second fundamental form

A plane containing the normal $\underset{\sim}{n}^{(\nu)}$ to the surface at a point P cuts the surface in different curves when rotated about the normal as an axis. Each curve so generated belongs both the surface and to the space E^3. A study of curvature properties of these curves reveals the curvature properties of the surfaces in which they lie. We decompose the curvature vector $\hat{\underset{\sim}{k}}$ at P of C, defined in Eq. (5), into a vector $\underset{\sim}{k}_n$ normal to the surface and a vector $\underset{\sim}{k}_g$ tangential to the surface as shown below:



Thus

$$\hat{\underset{\sim}{k}}^{(\nu)} = \underset{\sim}{k}_n^{(\nu)} + \underset{\sim}{k}_g^{(\nu)} \tag{16}$$

The vector $\underset{\sim}{k}_n$ is the normal curvature vector at the point P, and is given by

$$\underset{\sim}{k}_n^{(\nu)} = \underset{\sim}{n}^{(\nu)} \; k_n^{(\nu)} \tag{17}$$

where $k_n^{(\nu)}$ is its magnitude. To find an expression for $k_n^{(\nu)}$ we consider the equation

$$\underset{\sim}{n}^{(\nu)} \cdot \underset{\sim}{t} = 0$$

and differentiate it with respect to s (the arc length along the curve C) to have

$$k_n^{(\nu)} = - \frac{d\underset{\sim}{n}^{(\nu)} \cdot d\underset{\sim}{r}}{(ds)^2} \tag{18a}$$

Also, differentiating the equation

$$\underset{\sim}{n}^{(\nu)} \cdot \underset{\sim}{a}_\alpha = 0$$

ξ^{ν}

with respect to ξ^{β}, we get

$$(\mathbf{n}^{(\nu)})_{\xi^\beta} \cdot \mathbf{a}_\alpha = -\mathbf{n}^{(\nu)} \cdot (\mathbf{a}_\alpha)_{\xi^\beta} \tag{18b}$$

Further,

$$d\mathbf{n}^{(\nu)} = \sum_\alpha \frac{\partial \mathbf{n}^{(\nu)}}{\partial \xi^\alpha} d\xi^\alpha$$

$$d\mathbf{r} = \sum_\beta \frac{\partial \mathbf{r}}{\partial \xi^\beta} d\xi^\beta \tag{18c}$$

Thus using Eq. (18b) and (18c) in (18a), we get

$$k_n^{(\nu)} = \sum_{\alpha,\beta} b_{\alpha\beta} \frac{d\xi^\alpha d\xi^\beta}{(ds)^2} \tag{19}$$

where

$$b_{\alpha\beta} = \mathbf{n}^{(\nu)} \cdot \mathbf{r}_{\xi^\alpha \xi^\beta} \tag{20}$$

The two extreme values of $k_n^{(\nu)}$ are called the principal curvatures k_I and k_{II} and their sum is given by

$$k_I + k_{II} = \sum_{\alpha,\beta} g^{\alpha\beta} b_{\alpha\beta} \tag{21}$$

The form

$$\sum_{\alpha,\beta} b_{\alpha\beta} d\xi^\alpha d\xi^\beta \tag{22}$$

is called the second fundamental form.

3. Christoffel Symbols

Certain 3-index symbols, known as the Christoffel symbols, show up in a natural way when vectors or tensors are differentiated with respect to general coordinates introduced in a space. Here, by 'space' we mean a region in which arbitrary independent coordinates can be introduced; the number of independent coordinates determines the dimensionsion of the sapce. A space is termed Eulclidean when rectangular cartesian coordinates can be introduced in it on a global scale. Examples are 2D or 3D regions in a plane or in a rectangular box, respectively. It must, however, be pointed out that in an Euclidean space, besides rectangular cartesian coordinates, any general coordinate system can be introduced without disturbing the basic nature of the space itself. Since this book is mainly concerned with the general coordinate systems in either 2D or 3D Euclidean spaces, or to 2D surfaces embedded in a 3D space, we shall restrict our attention to the Christoffel symbols for space and for surfaces only.

A. Space Christoffel symbols

From the definition of the base vectors $\underline{a}_i$, we first note the following result. For any two indices i and k,

$$\frac{\partial \underline{a}_i}{\partial \xi^k} = \frac{\partial}{\partial \xi^k}\left(\frac{\partial \underline{r}}{\partial \xi^i}\right) = \frac{\partial}{\partial \xi^i}\left(\frac{\partial \underline{r}}{\partial \xi^k}\right)$$

Thus

$$\frac{\partial \underline{a}_i}{\partial \xi^k} = \frac{\partial \underline{a}_k}{\partial \xi^i} \tag{23}$$

We now select any three indices. say i,j,k, and consider the following three equations,

$$\frac{\partial g_{ij}}{\partial \xi^k} = \frac{\partial}{\partial \xi^k}(\underline{a}_i \cdot \underline{a}_j)$$

$$\frac{\partial g_{jk}}{\partial \xi^i} = \frac{\partial}{\partial \xi^i}(\underline{a}_j \cdot \underline{a}_k)$$

$$\frac{\partial g_{ik}}{\partial \xi^j} = \frac{\partial}{\partial \xi^j}(\underline{a}_i \cdot \underline{a}_k)$$

Adding the second and third equations, and subtracting the first equation, while using Eq. (23), we get

$$\frac{\partial \underline{a}_i}{\partial \xi^j} \cdot \underline{a}_k = [ij,k] \tag{24}$$

where

$$[ij,k] = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial \xi^i} + \frac{\partial g_{ik}}{\partial \xi^j} - \frac{\partial g_{ij}}{\partial \xi^k}\right) \tag{25}$$

is called the Christoffel symbol of the first kind.

Eq. (24) implies that

$$\frac{\partial \underset{\sim}{a}_i}{\partial \xi^j} = \sum_k [ij,k] \, \underset{\sim}{a}^{(k)} \tag{26}$$

Taking the dot product on both sides of Eq. (26) by $\underset{\sim}{a}^l$, we obtain

$$\frac{\partial a_i}{\partial \xi^j} \cdot a^l = \Gamma^l_{ij} \tag{27}$$

where

$$\Gamma^l_{ij} = \sum_k g^{kl} \, [ij,k] \tag{28}$$

is called the Christoffel symbol of the second kind.

Eq.(27) implies that

$$\frac{\partial \underset{\sim}{a}_i}{\partial \xi^j} = \sum_l \Gamma^l_{ij} \, a_l \tag{29}$$

It must be noted that both kinds of Christofiel symbols are symmetric in the first two indices, viz.,

$$[ij,k] = [ji,k], \qquad \Gamma^l_{ij} = \Gamma^l_{ji}$$

It is also easy to show, based on the definition of Γ^l_{ij} that

$$\sum_i \Gamma^i_{ij} = \frac{1}{2g} \frac{\partial g}{\partial \xi^j} \tag{30}$$

The Christoffel symbols Γ^l_{ij} can be computed by using the following expanded formulae:

$$\Gamma^l_{ij} = \sum_k \sum_n g^{kl} \frac{\partial x_n}{\partial \xi^k} \frac{\partial^2 x_n}{\partial \xi^i \xi^j} \tag{31}$$

where the indices l,i,j range from 1 to 3 in 3D, or from 1 to 2 in 2D.

B. Christoffel symbols in a surface

The Christoffel symbols, (25) and (28) are applicable both to 2D and 3D Euclidean spaces. In fact, if we take (25) and (28) as the definitions of some 3-index symbols without any consideration of an Euclidean space, then they are also applicable to an n-dimensional non-Euclidean space.

The Christoffel symbols for a 2D surface embedded in a 3D Euclidean space are defined exactly as for any other space. Since in a surface only two independent coordinates can be introduced, we again use the Greek indices to emphasize this point and write

$$[\alpha\beta,\delta] = \frac{1}{2}\left(\frac{\partial g_{\alpha\delta}}{\partial \xi^{\beta}} + \frac{\partial g_{\beta\delta}}{\partial \xi^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial \xi^{\delta}}\right) \tag{32}$$

$$\Gamma^{\sigma}_{\alpha\beta} = \sum_{\delta} g^{\delta\sigma}\,[\alpha\beta,\delta] \tag{33}$$

as the Christoffel symbols of the first and second kind respectively, of a surface. Here the indices assume only two values.

An important point to note here is that for a 2D space the metric coefficients g_{ij} do not depend on one of the cartesian coordinate, say z. On the other hand for a 2D space formed by a surface in 3D Euclidean space the metric coefficients appearing in (32) and (33) depend on all three cartesian coordinates.

Gauss indirectly introduced the definition of the Christoffel symobls by arguing that in a surface the base vectors $\underset{\sim}{r}_{\alpha}$, $\underset{\sim}{r}_{\beta}$ and the unit normal $\underset{\sim}{n}$ (Eq. (15)) form a triad of independent vectors. Thus any other vector in the surface can be presented as a linear combination of $\underset{\sim}{r}_{\alpha}$, $\underset{\sim}{r}_{\beta}$, $\underset{\sim}{n}$. Following this argument, the second derivative of the position vector $\underset{\sim}{r}$ can be expressed as

$$\underset{\sim}{r}_{\alpha\beta} = \sum_{\delta} \Gamma^{\delta}_{\alpha\beta}\, \underset{\sim}{r}_{\delta} + b_{\alpha\beta}\, \underset{\sim}{n} \tag{34}$$

which are called the formulae of Gauss. Thus, for a surface ξ^3 = constant in which ξ^1, ξ^2 are the current coordinates, Eq. (34) is written as

$$\underset{\sim}{r}_{\alpha\beta} = \sum_{\delta} \Gamma^{\delta}_{\alpha\beta}\, \underset{\sim}{r}_{\delta} + b_{\alpha\beta}\, \underset{\sim}{n}^{(\nu)} \tag{35}$$

where Eq. (35) represents the second derivatives $\underset{\sim}{r}_{\xi^1\xi^1}$, $\underset{\sim}{r}_{\xi^1\xi^2}$, $\underset{\sim}{r}_{\xi^2\xi^2}$.